**FP3 mark schemes from old P4, P5, P6 and FP1, FP2, FP3 papers (back to June 2002)**

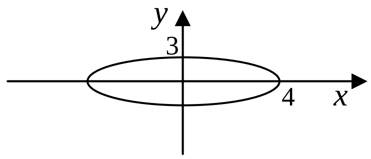
Please note that the following pages contain mark schemes for questions from past papers.

Where a question reference is marked with an asterisk (*), it is a partial version of the original.

The standard of the mark schemes is variable, depending on what we still have – many are scanned, some are handwritten and some are typed.

The questions are available on a separate document, originally sent with this one.

*This document was circulated by e-mail in March 2009; the mark schemes for questions 2 and 7 have since been removed (18.3.09) since they are not on the specification.*

| | | | |
|---|---|---|---|
| **1.** (*a*) |  Closed shape 3, 4 | B1 | **(1)** |
| (*b*) | $b^2 = a^2(1 - e^2)$ $\Rightarrow$ $9 = 16(1 - e^2)$ | M1 | |
| | $e = \frac{\sqrt{7}}{4}$ oe awrt 0.661 | A1 | **(2)** |
| (*c*) | Foci are at $(\pm ae\ , 0)$ use of *ae* | M1 | |
| | $(\sqrt{7}, 0)$ <u>and</u> $(-\sqrt{7}, 0)$ awrt 2.65, 0 is required, ft their *e* | A1 ft | **(2)** |
| | | | **(5 marks)** |

[P5 June 2002 Qn 1]

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| **3.** | $10\left(\frac{e^x+e^{-x}}{2}\right)+2\left(\frac{e^x-e^{-x}}{2}\right)=11$ | M1 |
| | $6e^{2x}-11x^x+4=0$ quadratic in $e^x$ | M1, A1 |
| | $(2e^x-1)(3e^x-4)=0$ | M1 |
| | $e^x$ = ½ and $^4/_3$ | A1 |
| | $x$ = ln ½ and ln $^4/_3$ | M1, A1 |
| | | **(7 marks)** |

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| **Alt 3.** | $10\cosh x+2\sinh x\equiv R\cosh(x+\alpha)$ | |
| | $R=\sqrt{96}$ <u>and</u> $\tan\alpha=\frac{1}{5}$ | M1, A1 |
| | $\cosh(x+\alpha)=\frac{11}{\sqrt{96}}$ | |
| | $x+\alpha=\ln\left[\frac{11}{\sqrt{96}}\pm\sqrt{\left(\frac{121}{96}\right)-1}\right]$ either | A1 |
| | $=\ln\frac{4}{\sqrt{6}}$ <u>and</u> $\ln\frac{\sqrt{6}}{4}$ both as single ln | A1 |
| | $x=\ln\frac{4}{\sqrt{6}}-½\ln\frac{3}{2},\ \ln\frac{\sqrt{6}}{4}-½\ln\frac{3}{2}$ | |
| | $=\ln\left(\frac{4}{\sqrt{6}}-\frac{\sqrt{2}}{\sqrt{3}}\right),\ \ln\left(\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{\sqrt{3}}\right)$ combine either into single ln. Dependent on first two Ms | M1 |
| | $x$ = ln ½ and ln $^4/_3$ | A1, A1 |
| | [One answer by alt. method gains max. M1A1M1M0M1A1A0] | |
| | | **(11 marks)** |

[P5 June 2002 Qn 3]

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| **4.** | (*a*) | $\int x^n \cos x \,\mathrm{d}x = x^n \sin x - n\int x^{n-1} \sin x \,\mathrm{d}x$ | M1, A1 |
| | | $= \ldots\ldots \quad -[nx^{n-1} \cos x + \int n(n-1)x^{n-2} \cos x \,\mathrm{d}x]$ | M1 |
| | | Using limits $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ (✱) cso | M1, A1 **(5)** |
| | (*b*) | $I_0 = \int_0^{\frac{\pi}{2}} \cos x \,\mathrm{d}x = [\sin x]_0^{\frac{\pi}{2}} = 1$ at any stage | B1 |
| | | $I_6 = \left(\frac{\pi}{2}\right)^6 - 30I_4$ | M1 |
| | | $= \left(\frac{\pi}{2}\right)^6 - 30\left(\left(\frac{\pi}{2}\right)^6 - 12I_2\right)$ | |
| | | $= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720I_0$ | M1 |
| | | Hence $I_6 = \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720$ cao | A1 **(4)** |
| | | | **(9 marks)** |

[P5 June 2002 Qn 4]

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| **5.** (*a*) | $y = \arctan 3x \quad \Rightarrow \quad \tan y = 3x$ | M1 | |
| | $\sec^2 y \dfrac{dy}{dx} = 3$ | A1 | |
| | $\dfrac{dy}{dx} = \dfrac{3}{1+\tan^2 y} = \dfrac{3}{1+9x^2}$ (✱) | M1, A1 | **(4)** |
| (*b*) | $\int 6x \arctan 3x \, dx = 3x^2 \arctan 3x - \int \dfrac{9x^2}{1+9x^2} dx$ | M1, A1 | |
| | $= \ldots\ldots \quad - \int \dfrac{1+9x^2-1}{1+9x^2} dx$ | M1 | |
| | $= \ldots\ldots \quad -x + \dfrac{1}{3}\arctan 3x$ | A1 | |
| | $\Big[ \quad \Big]_0^{\frac{\sqrt{3}}{3}} = \dfrac{\pi}{3} - \dfrac{\sqrt{3}}{3} + \dfrac{\pi}{9}$ | M1 | |
| | $= \dfrac{1}{9}(4\pi - 3\sqrt{3})$ (✱) cso | A1 | **(6)** |
| | | | **(10 marks)** |

[P5 June 2002 Qn 6]

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| **6.** (*a*) | $y = 2x^{½}$ , $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^{-½}$ | M1, A1 | |
| | $\int_{\dots}^{\dots} 2\pi y\left[1+\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^2\right]\mathrm{d}x = 4\pi\int_{\dots}^{\dots} x^{½}\left[1+\dfrac{1}{x}\right]^{½}\mathrm{d}x$ | M1 | |
| | $= 4\pi\int_0^1 \sqrt{1+x}\,\mathrm{d}x$ (✱) | A1 | **(4)** |
| (*b*) | $S = 4\pi\int_{\dots}^{\dots}\sqrt{1+x}\,\mathrm{d}x = \left[4\pi\dfrac{2}{3}(1+x)^{3/2}\right]_{(0)}^{(1)}$ | M1, A1 | |
| | $= \dfrac{8\pi}{3}\left(2^{3/2}-1\right)$ or any exact equivalent | A1 | **(3)** |
| (*c*) | $\int\left(1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{1}{2}}\mathrm{d}x = \int\left(1+\dfrac{1}{x}\right)^{\frac{1}{2}}\mathrm{d}x$ | M1 | |
| | $\int\sqrt{\dfrac{x+1}{x}}\,\mathrm{d}x$ | A1 | |
| | Using symmetry, $s = 2\int_0^1\sqrt{\dfrac{x+1}{x}}\,\mathrm{d}x$ (✱) | A1 | **(3)** |
| (*d*) | $x = \sinh^2\theta$, $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sinh\theta\cosh\theta$ oe | B1 | |
| | $I = 2\int\sqrt{\dfrac{1+\sinh^2\theta}{\sinh^2\theta}}.2\sinh\theta\cosh\theta\;\mathrm{d}\theta$ | M1 | |
| | $= 4\int\cosh^2\theta\;\mathrm{d}\theta$ | | |
| | $= 2\int(1+\cosh 2\theta)\;\mathrm{d}\theta$ | M1 | |
| | $= 2\theta + \sinh 2\theta$ | A1 | |
| | Limits are 0 and arsinh1 $(= \ln(1+\sqrt{2})$ | | |
| | $s = \left[2\theta + 2\sinh\theta\sqrt{1+\sinh^2\theta}\right]_0^{\text{arsinh}1}$ | | |
| | $= 2\,\text{arsinh}1 + 2\sqrt{(1+1^2)} = 2\left[\sqrt{2}+\ln(1+\sqrt{2})\right]$ | M1 A1 | **(6)** |

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| **6. (d) Alt** | The last four marks can be gained: | |
| | $I = 4\int \left(\frac{e^{\theta} + e^{-\theta}}{2}\right)^2 d\theta = \int (e^{2\theta} + 2 + e^{-2\theta}) d\theta$ | M1 |
| | $= \frac{e^{2\theta}}{2} + 2\theta - \frac{e^{-2\theta}}{2}$ | A1 |
| | $s = 2\,\text{arsinh}\,1 + \frac{1}{2}\left[\left(1+\sqrt{2}\right)^2 - \frac{1}{\left(1+\sqrt{2}\right)^2}\right]$ | |
| | $= \ldots. \quad + \frac{1}{2}\left[1 + 2 + 2\sqrt{2} - \frac{1}{3+2\sqrt{2}} \cdot \frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right]$ | |
| | $= 2\ln(1+\sqrt{2}) + ½(3 + 2\sqrt{2} - 3 + 2\sqrt{2}) = 2[\sqrt{2} + \ln(1+\sqrt{2})]$ (✱) | M1, A1 |
| **6. (d) Alt** | The last two marks may be gained by substituting back to the variable *x* | |
| | $s = \left[2\theta + \sinh 2\theta\right]_{\ldots}^{\ldots} = \left[2\theta + 2\sinh\theta\cosh\theta\right]_{\ldots}^{\ldots}$ | |
| | $= \left[2\text{arsinh}\sqrt{x} + 2\sqrt{x}\sqrt{1+x}\right]_0^1$ | |
| | $= 2\text{arsinh}1 + 2\sqrt{2} = 2\ln(1+\sqrt{2}) = 2\sqrt{2}$ | |
| | $= 2\left[\sqrt{2} + \ln(1+\sqrt{2})\right]$ (✱) | M1, A1 |

[P5 June 2002 Qn 8]

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| **8.** (*a*) | $\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}6\\-6\\3\end{pmatrix}=3\begin{pmatrix}2\\-2\\1\end{pmatrix}$, $\therefore$ eigenvalue is 3 | M1A1, A1 **(3)** |
| (*b*) | Either $\begin{vmatrix}-8&0&4\\0&-4&4\\4&4&-6\end{vmatrix}=-8(24-16)+4(16)=-64+64=0$ | M1 A1 **(2)** |
| Alt(*b*) | or $\begin{vmatrix}1-\lambda&0&4\\0&5-\lambda&4\\4&4&3-\lambda\end{vmatrix}=0$<br>$\Rightarrow(1-\lambda)(5-\lambda)(3-\lambda)-16(1-\lambda)-16(5-\lambda)=0$<br>$\Rightarrow(3-\lambda)(\lambda-9)(\lambda+3)=0\ \Rightarrow\lambda$ is an eigenvalue | M1 A1 |
| | $\begin{pmatrix}x\\y\\z\end{pmatrix}$ eigenvector $\Rightarrow$ $x+4z=9x$, $5y+4z=9y$, $4x+4y+3z+9z$ | M1 |
| | At least two of these equations<br>Attempt to solve $z=2x$, $z=y$, $2x+2y=3z$ | A1 |
| | $\therefore\begin{pmatrix}1\\2\\2\end{pmatrix}$ | A1 **(3)** |
| (*c*) | Make e.vectors unit to obtain $\mathbf{P}=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&-\frac{2}{3}\end{pmatrix}$ columns in any order | M1, A1ft |
| | $\mathbf{D}=\begin{pmatrix}3&0&0\\0&9&0\\0&0&\lambda_3\end{pmatrix}$, where $\lambda_3=-3$, **P** and **D** consistent | M1, A1, B1 **(5)** |
| **Alt** | $\mathbf{P}=\begin{pmatrix}2&1&2\\-2&2&1\\1&2&-2\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}27&0&0\\0&81&0\\0&0&-27\end{pmatrix}$, **P** and **D** consistent | M1A1ft, M1A1, B1 |

[P6 June 2002 Qn 5]

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| **9.** (*a*) | $\overrightarrow{AB} = 5\mathbf{i} + 3\mathbf{j}$  $\overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ or $\overrightarrow{BC} = -2\mathbf{i} - \mathbf{j} - \mathbf{k}$ | | |
| | $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 0 \\ 3 & 2 & -1 \end{vmatrix} = -3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ | M1, A1 | |
| | $\therefore \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ | B1 ft | **(3)** |
| (*b*) | Volume = $\frac{1}{6}\overrightarrow{AD}.(\overrightarrow{AB} \times \overrightarrow{AC})$  $\overrightarrow{AD} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ | B1 | |
| | $= {}^{1}/_{6}(2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}).(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ | M1 | |
| | $= {}^{11}/_{6}$ | A1 | **(3)** |
| (*c*) | $\mathbf{r}.(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = (2\mathbf{i} + \mathbf{j}).(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ | M1, A1 ft | |
| | $= -1$ | A1 | **(3)** |
| (*d*) | $[\mathbf{i}.(1 - 3\lambda) + \mathbf{j}(2 + 5\lambda) + \mathbf{k}(1 + \lambda)].(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = -1$ | M1, A1ft | |
| | $-3 + 9\lambda + 10 + 25\lambda + 3 + \lambda = -1$ | | |
| | $35\lambda + 10 = -1 \quad \Rightarrow \quad \lambda = -\frac{11}{35}$ | M1 | |
| | $\therefore$ E is $\left(\frac{68}{35}, \frac{15}{35}, \frac{94}{35}\right)$ | A1 | **(4)** |
| (*e*) | Distance = $-\frac{11}{35}\left\|-3i + 5j + k\right\| = \frac{11\sqrt{35}}{35}$  (✱) | M1 A1 | **(2)** |
| (*f*) | $\lambda = 2 \times \left(-\frac{11}{35}\right) = -\frac{22}{35}$ | B1 | |
| | $\mathbf{r}_{D'} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + -\frac{22}{35}(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ | M1 | |
| | D′ is $\left(\frac{101}{35}, -\frac{40}{35}, \frac{83}{35}\right)$ | A1 | **(3)** |
| | | | **(18 marks)** |

[P6 June 2002 Qn 7]

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| **10.** | $4\left(\frac{e^x + e^{-x}}{2}\right) + \frac{e^x - e^{-x}}{2} = 8$ | M1 |
| | $5e^{2x} - 16e^x + 3 = 0$ | M1 A1 |
| | $(5e^x - 1)(e^x - 3) = 0$ | A1 |
| | $e^x = \frac{1}{5}, 3$ | |
| | $x = \ln(\frac{1}{5}), \ln 3$ accept – ln 5 | M1 A1 (6) |
| | | **(6 marks)** |

[P5 June 2003 Qn 1]

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| **11** (*a*) | $y = \text{artanh } x$ | |
| | $\tanh y = x$ | |
| | $\text{sech}^2 y \frac{dy}{dx} = 1$ | M1 A1 |
| | $\frac{dy}{dx} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}$ (*) | A1 (3) |
| | cso | |
| (*b*) | $\int 1.\text{artanh } x \, dx = x \text{ artanh } x - \int \frac{x}{1 - x^2} \, dx$ | M1 A1 |
| | $= x \text{ artanh } x + \frac{1}{2} \ln(1 - x^2) \; (+ c)$ | M1 A1 (4) |
| | | **(7 marks)** |
| Alt. | $\frac{x}{1 - x^2} \equiv \frac{1}{2}\left[\frac{1}{1 - x} - \frac{1}{1 + x}\right]$ | |
| | $\int \frac{x}{1 - x^2} \, dx = -\frac{1}{2} \ln(1 - x) - \frac{1}{2} \ln(1 + x)$ | |
| | This is acceptable (with the rest correct) for final M1 A1 | |

[P5 June 2003 Qn 2]

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| **12.** | $\int \frac{10}{\sqrt{4x^2+9}}\,dx = \frac{1}{2}\int \frac{10}{\sqrt{x^2+\frac{9}{4}}}\,dx$ | M1 |
| | $= \frac{10}{2}\text{arsinh}\left(\frac{2x}{3}\right) \quad \left(= 5\ln\left[\frac{2x}{3}+\sqrt{\frac{4x^2}{9}+1}\right]\right)$ | M1 A1 |
| | $[\ ]_0^5 = 5\text{ arsinh}\frac{10}{3} \quad \left(= 5\ln\left(\frac{10}{3}+\sqrt{\frac{109}{9}}\right) \approx 9.594\right)$ | M1 A1 ft |
| | ft on 5 | |
| | Area = 9.594 × 100 = 960 ($m^2$) | M1 A1 |
| | | **(7 marks)** |
| | Using a substitution | |
| | (i) $2x = 3\sinh\theta$; $2\,dx = 3\cosh\theta\,d\theta$ | |
| | $\int \frac{10}{\sqrt{4x^2+9}}\,dx = \int \frac{10}{3\cosh\theta}\times\frac{3}{2}\cosh\theta\,d\theta$ complete subs. | M1 |
| | $= 5\int d\theta = 5\text{ arsinh}\frac{2x}{3}$ | M1 A1 |
| | then as before, | |
| | or changing limits to 0 and arsinh $\frac{10}{3}$ (or $\ln\left(\frac{10}{3}+\sqrt{\frac{109}{9}}\right)$) can gain this A1 | |
| | (ii) $2x = 3\tan\theta$; $2\,dx = 3\sec^2\theta\,d\theta$ | |
| | $\int \frac{10}{\sqrt{4x^2+9}}\,dx = \int \frac{10}{\sqrt{9\tan^2\theta+9}}\times\frac{3}{2}\sec^2\theta\,d\theta$ | M1 |
| | $= 5\int \sec\theta\,d\theta = 5\ln(\sec\theta+\tan\theta)$ | M1 |
| | Limits are 0 and arctan $\frac{10}{3}$ | A1 |
| | $[\ ]_0^{\arctan\frac{10}{3}} = 5\ln\left(\sqrt{\frac{100}{9}+1}+\frac{10}{3}\right)$ etc | M1 A1ft |

[P5 June 2003 Qn 3]

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| **13.** | (*a*) | $\dfrac{dy}{dx} = \sinh \dfrac{x}{a}$ | B1 |
| | | $\int \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\ dx = \int \sqrt{1+\sinh^2\left(\dfrac{x}{a}\right)}\ dx = \int \cosh \dfrac{x}{a}\ dx = \sinh \dfrac{x}{a}$ | M1, A1 |
| | | Length = $2\left[a \sinh \dfrac{x}{a}\right]_0^{ka} = 2a \sinh k$ (*) | M1 A1 (5) |
| | (*b*) | $2a \sinh k = 8a$ | |
| | | $\sinh k = 4$ | B1 |
| | | $x = ka = a \text{ arsinh } 4 = a \ln (4 + \sqrt{17})$ | B1 |
| | | $y = a \cosh \dfrac{ka}{a} = a\sqrt{(1 + \sinh^2 k)} = a\sqrt{17}$ | M1 A1 (4) |
| | | | **(9 marks)** |

[P5 June 2003 Qn 5]

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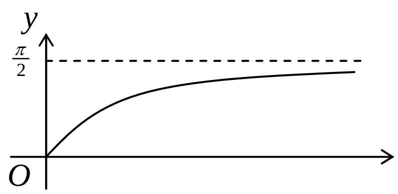
| **14.** (*a*) | $\sec y = e^x$ | | B1 |
| | $\sec y \tan y \dfrac{dy}{dx} = e^x \quad (= \sec y)$ | | M1 A1 |
| | $\dfrac{dy}{dx} = \dfrac{e^x}{\sec y \tan y} = \dfrac{1}{\sqrt{\sec^2 y - 1}} = \dfrac{1}{\sqrt{e^{2x} - 1}} \quad (*)$ | cso | M1 A1 (5) |
| (*b*) |  | Shape, curve → (0, 0) | B1 |
| | | Asymptote, ($y =$ ) $\dfrac{\pi}{2}$ | B1 (2) |
| (*c*) | ($x = \ln 2$) $\quad y = \text{arcsec } 2 = \dfrac{\pi}{3}$ | | B1 |
| | $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4-1}} = \dfrac{1}{\sqrt{3}}$ | | B1 |
| | tangent is $\quad y - \dfrac{\pi}{3} = \dfrac{1}{\sqrt{3}}(x - \ln 2)$ | | M1 |
| | $x = 0, \quad y = \dfrac{\pi}{3} - \dfrac{1}{\sqrt{3}} \ln 2$ | exact answer only | A1 (4) |
| | | | **(11 marks)** |
| Alt to (*a*) | $\cos y = e^{-x}$ | | B1 |
| | $-\sin y \dfrac{dy}{dx} = \dfrac{e^{-x}}{\sqrt{1 - \cos^2 y}}$ | | M1 A1 |
| | $= \dfrac{e^{-x}}{\sqrt{1 - e^{-2x}}} = \dfrac{1}{\sqrt{e^{2x} - 1}} \quad (*)$ | cso | M1 A1 (5) |

[P5 June 2003 Qn 6]

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| **15.** | (*a*) | $I_n = \left[x^n e^x\right]_0^1 - n\int_0^1 x^{n-1} e^x \, dx = e - nI_{n-1}$ (*) cso | M1 A1 (2) |
| | (*b*) | $J_n = \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1} e^{-x} \, dx$ | M1 A1 |
| | | $= -e^{-1} + nJ_{n-1}$ | A1 (3) |
| | (*c*) | $J_2 = -e^{-1} + 2J_1$ | |
| | | $J_1 = -e^{-1} + J_0$ $J_2$ <u>and</u> $J_1$ | M1 |
| | | $= -e^{-1} + \int_0^1 e^{-x} \, dx$ | |
| | | $= -e^{-1} + (1 - e^{-1})$ $( = 1 - 2e^{-1})$ | A1 |
| | | $J_2 = -e^{-1} + 2(1 - 2e^{-1}) = 2 - \frac{5}{e}$ (*) | A1 (3) |
| | (*d*) | $\int_0^1 x^n \cosh x \, dx = \int_0^1 x^n \left(\frac{e^x + e^{-x}}{2}\right) dx = \frac{1}{2}(I_n + J_n)$ (*) | B1 (1) |
| | (*e*) | $I_2 = e - 2I_1 = e - 2(e - I_0) = 2I_0 - e$ | |
| | | $= 2\int_0^1 e^x \, dx - e = 2[e - 1] - e$ $( = e - 2)$ | M1 A1 |
| | | $\frac{1}{2}(I_2 + J_2) = \frac{1}{2}(e - 2 + 2 - \frac{5}{e}) = \frac{1}{2}(e - \frac{5}{e})$ | M1 A1 (4) |
| | | | **(13 marks)** |

[P5 June 2003 Qn 7]

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| **16.** (*a*) | $\dfrac{dx}{dt} = a \sec t \tan t, \quad \dfrac{dy}{dt} = b \sec^2 t$ | M1 A1 |
| | $\dfrac{dy}{dx} = \dfrac{b\sec^2 t}{a\sec t\tan t} \quad \left(= \dfrac{b}{a\sin t}\right)$ | M1 A1 |
| | gradient of normal is $-\dfrac{a\sin t}{b}$ | |
| | $y - b\tan t = -\dfrac{a\sin t}{b}(x - a\sec t)$ | M1 |
| | $ax\sin t + by = (a^2 + b^2)\tan t$ (*) cso | A1 (6) |
| (*b*) | $y = 0 \Rightarrow \quad x = \dfrac{(a^2+b^2)\tan t}{a\sin t} \quad \left(= \dfrac{a^2+b^2}{a\cos t}\right)$ | B1 |
| | $b^2 = a^2(e^2 - 1) \;\Rightarrow\; b^2 = \dfrac{5a^2}{4}$ | M1 |
| | $OS = ae$ and $OA = 3AS$ | M1 |
| | $a^2 + \dfrac{5a^2}{4} = 3a^2 \times \dfrac{3}{2} \times \cos t$ | |
| | $\cos t = \frac{1}{2}$ | M1 A1 |
| | $t = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ | A1 |
| | By symmetry or (as $OA = \left\|\dfrac{a^2+b^2}{a\cos t}\right\|$) $-\dfrac{a^2+b^2}{a\cos t} = 3ae$ | |
| | $t = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ | M1 A1 (8) |
| | | **(14 marks)** |
| Alt. to (*a*) | $\dfrac{2x}{a^2} - \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$ | M1 A1 |
| | $\dfrac{dy}{dx} = \dfrac{2b^2x}{2a^2y} = \dfrac{b^2a\sec t}{a^2b\tan t} \ldots$ | M1 A1 |
| | then as before | |

[P5 June 2003 Qn 8]

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|---|---|---|
| **17.** | $\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (\mathbf{b} \times \mathbf{c}) - (\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{a})$ | M1 A1 |
| | Using $\mathbf{a} \times \mathbf{c} = -\mathbf{c} \times \mathbf{a}$ or $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ or $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ | B1 |
| | $\lvert \overrightarrow{AB} \times \overrightarrow{AC} \rvert = AB.\ AC \sin\theta = 2 \times$ area of triangle, or equivalent | M1 |
| | Final result : $\frac{1}{2} \lvert \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} \rvert$ (*) | A1 cso **[5]** |

[P6 June 2003 Qn 1]

| | | |
|---|---|---|
| **18.** | (a) Deriving characteristic equation $(4 - \lambda)(-9 - \lambda) + 30 = 0$ | M1 A1 |
| | $\Rightarrow \lambda^2 + 5\lambda - 6 = 0 \Rightarrow (\lambda + 6)(\lambda - 1) = 0 \Rightarrow \lambda = -6, \lambda = 1$ | M1 A1 (4) |
| | (b) Stating, implying or showing $\lambda = 1$ associated with point invariant line. | B1 |
| | $\Rightarrow \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ | |
| | Equation is $4x - 5y = x \Rightarrow 3x - 5y = 0$ any equivalent form | M1 A1 (3) **[7]** |

[P6 June 2003 Qn 3]

| | | | |
|---|---|---|---|
| **19.** | (a) Det $\mathbf{A} = 3(u-3) - (u-5) - (3-5) = 2u - 2 \quad [= 2(u-1)]$ (*) | M1 A1 | (2) |
| | (b) Cofactors $\begin{pmatrix} u-3 & 5-u & -2 \\ -(u+3) & 3u+5 & -4 \\ 2 & -4 & 2 \end{pmatrix}$ (– 1 A mark for each term wrong) | M1 A3 | |
| | $\mathbf{A}^{-1} = \frac{1}{2(u-1)}\begin{pmatrix} u-3 & -(u+3) & 2 \\ 5-u & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$ | M1 A1 ft | (6) |
| | (c) $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$, Using $u = 6$: $\frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 12 \\ -4 \\ 2 \end{pmatrix}$ | M1 | |
| | $a = \frac{6}{5}$, $b = -\frac{2}{5}$ $c = \frac{1}{5}$ (One correct A1, other 2 correct A1) | A1, A1 | (3) |
| | [ Algebraic approach: Finding one value M1 A1, other two A1 ] | | |
| | | | **[11]** |

[P6 June 2003 Qn 6]

| | | |
|---|---|---|
| **20.** | (a) Normal to plane is $(-\mathbf{i}+5\mathbf{j}+3\mathbf{k})$ | B1 |
| | Equation of plane: $\mathbf{r}.(-\mathbf{i}+5\mathbf{j}+3\mathbf{k}) = (\mathbf{i}+2\mathbf{j}-\mathbf{k}).(-\mathbf{i}+5\mathbf{j}+3\mathbf{k})$ | M1 |
| | $\Rightarrow\ -x+5y+3z = -1+10-3 = 6$ or equivalent (*) | M1 A1 (4) |
| | [If vector equation of plane is by-passed, then B1 M2 A1 ] | |
| | (b) $\frac{1}{\sqrt{35}}$ | B1 |
| | $\lvert 6-(\mathbf{i}+2\mathbf{j}+2\mathbf{k}).(-\mathbf{i}+5\mathbf{j}+3\mathbf{k})\rvert$ | M1 A1 |
| | or $\lvert\overrightarrow{PQ}.(-\mathbf{i}+5\mathbf{j}+3\mathbf{k})\rvert = \lvert 3\mathbf{k}.(-\mathbf{i}+5\mathbf{j}+3\mathbf{k})\rvert$ | |
| | Distance = $\frac{9}{\sqrt{35}}$ or a.w.r.t 1.52 | A1 (4) |
| | (c) Direction of one line in plane = $(-\mathbf{i}+5\mathbf{j}+3\mathbf{k})$ | M1 |
| | Direction of another line in plane = $(3\mathbf{i}-2\mathbf{k})-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$ | M1 |
| | $\Rightarrow \mathbf{r} = (\mathbf{i}+2\mathbf{j}+2\mathbf{k})+s(-\mathbf{i}+5\mathbf{j}+3\mathbf{k})+t(2\mathbf{i}-2\mathbf{j}-4\mathbf{k})$ | M1 A1 (4) |
| | or $(3\mathbf{i}-2\mathbf{k})+s(-\mathbf{i}+5\mathbf{j}+3\mathbf{k})+t(2\mathbf{i}-2\mathbf{j}-4\mathbf{k})$ | |
| | | **[12]** |

[P6 June 2003 Qn 7]

| | | | |
|---|---|---|---|
| **21.** | (*a*) | $\cosh^2 x - \sinh^2 x = \frac{1}{4}(e^x+e^{-x})^2 - \frac{1}{4}(e^x-e^{-x})^2$ | **M1** |
| | | $= \frac{1}{4}(e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x})$ | **A1** |
| | | $=1$ * | **A1** **(3)** |
| | (*b*) | $\frac{1}{\sinh x} - 2\frac{\cosh x}{\sinh x} = 2 \qquad \therefore 1-(e^x+e^{-x)} = e^x - e^{-x}$ | **M1** |
| | | $\therefore 2e^x = 1$ | **A1** |
| | | Make *x* the subject of the formula, $x = \ln(\frac{1}{2}) = -\ln 2$ | **M1, A1** (4) |

[P5 June 2004 Qn 1]

| | | | |
|---|---|---|---|
| **22.** | ($a$) | $a = 2, \quad b = 1, \quad c = 16$ | **B1, B1, B1** **(3)** |
| | ($b$) | $\int_{-0.5}^{1.5} \frac{1}{(2x+1)^2+16} \mathrm{d}x$ | **M1** |
| | | $= \left[ \frac{1}{8} \arctan\left( \frac{2x+1}{4} \right) \right]_{-0.5}^{1.5}$ | **M1 A1** |
| | | $= \frac{\pi}{32}$ | **B1** **(4)** |

[P5 June 2004 Qn 2]

| | | | | |
|---|---|---|---|---|
| **23.** | ($a$) | As $4 = 9(1-e^2)$, $\quad \therefore e^2 = \frac{5}{9}$ | | **M1, A1** |
| | | Uses *ae* to obtain that the foci are at $(\pm\sqrt{5}, 0)$ | | **M1 A1** **(4)** |
| | ($b$) | $PS + PS' = e(PM + PM')$ | M1 for single statement e.g, PS = ePM | **M1** |
| | | $= e \times \frac{2a}{e}$ | M1 needs complete method | **M1** |
| | | $= 2a = 6$ | | **A1** **(3)** |

[P5 June 2004 Qn 3]

| | | | |
|---|---|---|---|
| **24.** | (*a*) | Using product rule $\frac{dy}{dx} = (n-1)\sinh^{n-2} x \cosh^2 x + \sinh^n x$ | M1 |
| | | Using $\cosh^2 x = 1 + \sinh^2 x$ in derived expression | M1 |
| | | to obtain $\frac{dy}{dx} = (n-1)\sinh^{n-2} x(1+\sinh^2 x) + \sinh^n x$ | |
| | | and $\frac{dy}{dx} = (n-1)\sinh^{n-2} x + n\sinh^n x$ * | A1 (3) |
| | (*b*) | $[\sinh^{n-1} x \cosh x]_0^{ar\sinh 1} = \int_0^{ar\sinh 1} (n-1)\sinh^{n-2} x\mathrm{d}x + \int_0^{ar\sinh 1} n\sinh^n x\mathrm{d}x$ | M1 |
| | | So $\cosh(ar\sinh 1) = (n-1)I_{n-2} + nI_n$ | A1 (2) |
| | | If sinh α =1 then cosh α = $\sqrt{1+\sinh^2\alpha} = \sqrt{2}$ | |
| | | $\therefore nI_n = \sqrt{2} - (n-1)I_{n-2}$ * | |
| | | **OR** | |
| | | $\int_0^{ar\sinh 1} \sinh^{n-1} x \sinh x\mathrm{d}x = \left[\sinh^{n-1} x \cosh x\right]_0^{ar\sinh 1} - (n-1)\int_0^{ar\sinh 1} \cosh^2 x \sinh^{n-2} x\mathrm{d}x$ | |
| | | *and use* $\cosh^2 x = 1 + \sinh^2 x$ | M1 |
| | | *collect* $I_n + (n-1)I_n$ *to obtain* $nI_n = \sqrt{2} - (n-1)I_{n-2}$ * | A1 (2) |
| | (*c*) | $I_0 = \mathrm{ar}\sinh 1$ | B1 |
| | | $2I_2 = \sqrt{2} - I_0$ | M1 |
| | | $4I_4 = \sqrt{2} - 3I_2$ and use with previous results to obtain… | M1 |
| | | $= \frac{1}{8}(3\mathrm{ar}\sinh 1 - \sqrt{2})$ = 0.154 (either answer acceptable) | A1 (4) |

[P5 June 2004 Qn 5]

| | | | |
|---|---|---|---|
| **25.** | (*a*) | $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$ | B1 |
| | | $s = \int\sqrt{(9a^2(c^4s^2 + s^4c^2))}d\theta$ | M1 |
| | | $= 3a\int\sqrt{c^2s^2}d\theta$ | M1 |
| | | $= 3a\int\cos\theta\sin\theta d\theta$ | A1 |
| | | Total length = $4\times\frac{3a}{2}[\sin^2\theta]_0^{\frac{\pi}{2}}$ | M1 |
| | | $=6a$ | M1 A1 (7) |
| | (*b*) | $A = 2\pi\int a\sin^3\theta\times 3a\cos\theta\sin\theta d\theta$ | M1 A1 |
| | | $= 6\pi a^2\int\sin^4\theta\cos\theta d\theta$ | |
| | | $= \frac{6\pi a^2}{5}\left[\sin^5\theta\right]_0^{\frac{\pi}{2}}\times 2$ | M1 |
| | | $= \frac{12\pi a^2}{5}$ | M1 A1 (5) |

[P5 June 2004 Qn 7]

| | | | |
|---|---|---|---|
| **26.** | (*a*) | $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 3 \\ -4 \end{pmatrix}$ $\quad \overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ | M1 |
| | | $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ A1: One value correct, A1: All correct | M1 A1 A1 (4) |
| | (*b*) | $\mathbf{r}.\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}.\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 3 - 4 + 8$ $\quad \mathbf{r}.\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 7$ | M1 A1ft (2) |
| | (*c*) | $\overrightarrow{AD}\,.\,\overrightarrow{AB} \times \overrightarrow{AC}$ (Attempt suitable triple scalar product) | M1 |
| | | $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ (if using *AD*) | B1 |
| | | Volume = $\frac{1}{6}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}.\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \frac{1}{6}(2 + 12 - 2) = 2$ | M1 A1(cso)(4) |
| | | | **10** |

[P6 June 2004 Qn 3]

**27.** $\mathbf{MM^T} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix}\begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$

(a) $\begin{pmatrix} 1 & 4 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ p \end{pmatrix} = 0 \Rightarrow p = 3$ — M1 A1 (2)

(b) $\begin{pmatrix} 1 & 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = k \Rightarrow k = 18$ (ft on their $p$, if used) — M1 A1ft (2)

(c) 2 equations: $a + 4b - c = 0$ $3a + 3c = 0$ — M1

$a$ and $b$ in terms of $c$ (or equiv.): $a = -c$ $b = \frac{1}{2}c$ (ft on their $p$) — M1 A1ft

*Using* $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 18$ $(a^2 + b^2 + c^2 = 18)$**:** a = 2√2, b = –√2, c = –2√2 — M1 A2(1,0) (6)

(d) $|\det \mathbf{M}| = |(3\sqrt{2}) - 4(-12\sqrt{2}) - 1(-3\sqrt{2})| = 54\sqrt{2}$ — M1 A1(cso) (2)

**12**

Alternatives:

(c) Require $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ parallel to $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, $= \begin{pmatrix} 12 \\ -6 \\ -12 \end{pmatrix}$ — M1, M1 A1

(Then as in main scheme, scaling to give *a*, *b* and *c*.) — M1 A2(1,0) (6)

(d) $\det (\mathbf{MM^T}) = 18^3$, $\det \mathbf{M} = \det \mathbf{M^T}$, $|\det \mathbf{M}| = 18\sqrt{18}$ $(=54\sqrt{2})$ — M1 A1 (2)

[P6 June 2004 Qn 5]

**28.** (*a*) det **A** = 0 $(3-\lambda)^2 - 1 = 0$ M1

$\lambda^2 - 6\lambda + 8 = 0 \quad (\lambda - 2)(\lambda - 4) = 0 \quad \lambda = 2,\ \lambda = 4$ A1

$\lambda = 2: \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad x + y = 0, \quad$ Eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (or equiv.) M1 A1

$\lambda = 4: \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad -x + y = 0, \quad$ Eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or equiv.) A1 (5)

(*b*) $\mathbf{P} = k\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ M: eigenvectors as columns, $k = \frac{1}{\sqrt{2}}$ M1, A1

$\left\{ \mathbf{P}^{-1} = \mathbf{P}^{\mathbf{T}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}$

$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ M1, M1 A1 (5)

(*c*) 1. Rotation of $\frac{\pi}{4}$ clockwise (about (0, 0)).

2. Stretch, × 4 parallel to *x*-axis, × 2 parallel to *y*-axis.

3. Rotation of $\frac{\pi}{4}$ anticlockwise (about (0, 0)).

1. and 3. both rotation, or both reflection. M1

Correct angles, opposite sense or correct lines (reflection). A1

Stretch. B1

All correct, including order. A1 (4)

**14**

[P6 June 2004 Qn 6]

| | | | |
|---|---|---|---|
| **29.** | (a) $\int \frac{1+x}{\sqrt{(1-4x^2)}}dx = \int \frac{1}{\sqrt{(1-4x^2)}}dx + \int \frac{x}{\sqrt{(1-4x^2)}}dx$ | M1 | |
| | $\frac{1}{2}\arcsin 2x, + \; 2\times\frac{-1}{8}\sqrt{(1-4x^2)}$ one correct | M1 A1,M1 | |
| | $= \frac{1}{2}\arcsin 2x - \frac{1}{4}\sqrt{(1-4x^2)}\ (+C)$ | A1 | (5) |
| | Alternative Let $x = \frac{1}{2}\sin\theta$, $\int \frac{1+0.5\sin\theta}{\cos\theta}\times\frac{1}{2}\cos\theta d\theta$ M1 | | |
| | $= \frac{1}{2}\theta - \frac{1}{4}\cos\theta\ (+C)$, $= \frac{1}{2}\arcsin 2x - \frac{1}{4}\sqrt{(1-4x^2)}\ (+C)$ M1A1, M1 A1 | | |
| | (b) $\int_0^{0.3} \frac{1+x}{\sqrt{(1-4x^2)}}dx = 0.5\arcsin 0.6 - 0.25\sqrt{0.64} + 0.25 = 0.372$ | M1 A1 | (2) |
| | | | (7) |

[FP2/P5 June 2005 Qn 1]

| | | | |
|---|---|---|---|
| **30.** | (a) $\cosh 2x = \frac{e^{2x}+e^{-2x}}{2} = \frac{e^{2\ln k}+e^{-2\ln k}}{2}$ (or use $e^x = k$) | M1 | |
| | $= \frac{k^2+k^{-2}}{2} = \frac{k^4+1}{2k^2}$ (*) | M1 A1 | (3) |
| | (b) $f'(x) = p - 2\text{sech}^2 2x$ | M1 A1 | |
| | For $x = \ln 2$, $\cosh 2x = \frac{2^4+1}{2\times 2^2} = \frac{17}{8}$ | B1 | |
| | $p - \frac{2}{\cosh^2 2x} = 0$, $p = 2\times\frac{64}{289} = \frac{128}{289}$ | A1 | (4) |
| | | | (7) |

[FP2/P5 June 2005 Qn 2]

| | | |
|---|---|---|
| **31.** | $\frac{dx}{dt} = -3a\cos^2 t \sin t \quad \frac{dy}{dt} = 3a\sin^2 t \cos t$ | M1 A1 |
| | Area = $2\pi \int a\sin^3 t\sqrt{9a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t)}\,dt$ | M1 A1 |
| | $= 6\pi a^2 \int \sin^3 t \sin t \cos t\,dt = 6\pi a^2\left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = \frac{6\pi a^2}{5}$ | M1 A1 A1 (7) |
| | | (7) |

[FP2/P5 June 2005 Qn 3]

| | | | |
|---|---|---|---|
| **32.** | (a) $I_n = \frac{1}{2}x^n e^{2x} - \frac{n}{2}\int x^{n-1}e^{2x}dx, \quad I_n = \frac{1}{2}(x^n e^{2x} - nI_{n-1})$ (*) | | M1 A1 A1 (3) |
| | (b) $\int_0^1 x^2 e^{2x}dx = I_2 = \left[\frac{1}{2}x^2 e^{2x}\right]_0^1 - I_1 = \frac{1}{2}e^2 - I_1$ | one correct start | M1 |
| | $I_1 = \left[\frac{1}{2}xe^{2x}\right]_0^1 - \frac{I_0}{2} = \frac{1}{2}e^2 - \frac{1}{2}I_0$ | linking all three | M1 A1 |
| | $I_0 = \int_0^1 e^{2x}dx = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{e^2}{2} - \frac{1}{2}$ | | |
| | $I_2 = \frac{e^2}{2} - \left(\frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)\right) = \frac{1}{4}(e^2 - 1)$ | use of limits | M1<br>A1 (5) |
| | | | (8) |

[FP2/P5 June 2005 Qn 4]

| | | | |
|---|---|---|---|
| **33.** | $\int x\text{arcosh}x\,dx = \frac{x^2}{2}\text{arcosh}x - \int\frac{x^2}{2\sqrt{x^2-1}}dx$ | | M1 A1 |
| | $\left[\frac{x^2}{2}\text{arcosh}x\right]_1^2 = 2\text{ arcosh}2$ | | |
| | Let $x = \cosh\theta$ $\int\frac{\cosh^2\theta}{2\sinh\theta}\sinh\theta\,d\theta$ | | M1 A1 |
| | $= \int\frac{\cosh^2\theta}{2}d\theta = \int\frac{1+\cosh 2\theta}{4}d\theta, = \frac{\theta}{4} + \frac{\sinh 2\theta}{8}$ | | M1,M1 A1 |
| | $= \left[\frac{\theta}{4} + \frac{\sinh 2\theta}{8}\right]_0^{\text{arcosh}2} = \frac{1}{4}\text{arcosh}2 + \frac{2\times\sqrt{3}\times 2}{8}$ | limits | M1 A1 |
| | $\text{Area} = \frac{7}{4}\text{arcosh }2 - \frac{\sqrt{3}}{2} = \frac{7}{4}\ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}$ | (*) | A1 (10) |
| | | | **(10)** |

[FP2/P5 June 2005 Qn 6]

| | | | | |
|---|---|---|---|---|
| **34.** | (a) $\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right)=\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\times\frac{1+\sqrt{(1-x^2)}}{1+\sqrt{(1-x^2)}}\right)$ | | M1 | |
| | $=\ln\left(\frac{1-(1-x^2)}{x\left(1+\sqrt{(1-x^2)}\right)}\right)=-\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right)$ | (*) | M1 A1 | (3) |
| | (b) Let $y=\text{arsech}\,x$ $\quad$ $\text{sech}\,y=\frac{2}{e^y+e^{-y}}$ | | B1 | |
| | $xe^y+xe^{-y}=2 \quad xe^{2y}-2e^y+x=0$ | | M1 | |
| | $e^y=\frac{2\pm\sqrt{4-4x^2}}{2x}=\frac{1\pm\sqrt{1-x^2}}{x}$ | | M1 A1 | |
| | $y=\ln\left(\frac{1\pm\sqrt{1-x^2}}{x}\right)=(\pm)\ \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ | (*) | A1 | (5) |
| | (c) $3\left(1-\text{sech}^2 x\right)-4\text{sech}x+1=0$ | | M1 | |
| | $\left(3\text{sech}x-2\right)\left(\text{sech}x+2\right)=0 \quad \text{sech}x=\frac{2}{3}$ | | M1 A1 | |
| | $x=\pm\ln\left(\frac{3}{2}\left(1+\sqrt{\frac{5}{9}}\right)\right)=\pm\ln\left(\frac{3+\sqrt{5}}{2}\right)$ | | M1 A1 | (5) |
| | | | | **(13)** |

[FP2/P5 June 2005 Qn 8]

| | | |
|---|---|---|
| **35.** | (a) (i) $\mathbf{b} \times \mathbf{a}$ is perpendicular to $\mathbf{a}$ (and $\mathbf{b}$) | B1 |
| | $\mathbf{a}.\,\mathbf{b} \times \mathbf{a} = \lvert\mathbf{a}\rvert\,\lvert\mathbf{b} \times \mathbf{a}\rvert \cos 90° = 0$ or equivalent | B1 (2) |
| | (ii) $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Rightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$ | M1 |
| | As $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{c}$, | |
| | $\mathbf{a}$ is parallel to $(\mathbf{b} - \mathbf{c})$, so $\mathbf{b} - \mathbf{c} = \lambda\,\mathbf{a}$ | A1 (2) |
| | (b) (i) If $\mathbf{A}$ non-singular, then $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}$ (*)AG | M1A1 (2) |
| | (ii) $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$ | B1 |
| | Set $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$ and finding two equations | M1 |
| | Any non-zero values of $a$, $b$, $c$ and $d$ such that $a + 2c = 1$ and $b + 2d = 7$. | A1 (3) |
| | | **[9]** |

[FP3/P6 June 2005 Qn 2]

| | | |
|---|---|---|
| **36.** | (a) Normal to plane is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 6\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ (or any multiple) | M1A1 (2) |
| | (b) Equation of plane is $6x + y - 4z = d$ | M1 |
| | Substituting appropriate point in equation to give $6x + y - 4z = 16$ [e.g. (1, 6, –1), (3, –2, 0), (3, 6, 2) etc.] | A1 (2) |
| | (c) $p = -2$ | B1 (1) |
| | (d) Direction of line is perpendicular to both normals | |
| | $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 6 & 1 & -4 \end{vmatrix} = -9\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$ | M1 |
| | [Planes are: $6x + y - 4z = 16$, $x + 2y + z = 2$] | |
| | Finding a point on line | M1A1 |
| | **a** and **b** identified | M1 |
| | Any correct equation of correct form e.g. $\left[ r - \begin{pmatrix} -3 \\ 6 \\ -7 \end{pmatrix} \right] \times \begin{pmatrix} 9 \\ -10 \\ 11 \end{pmatrix} = \mathbf{0}$. | A1 (5) **[10]** |
| | *Alternative: Using equations of planes to find general point on line* | |
| | Using equations of planes to form any two of $10x + 9y = 24$, $11x - 9z = 30$, $11y + 10z = -4$ — M1 | |
| | Putting in parametric form — M1 | |
| | e.g. $\left( \lambda, \frac{24 - 10\lambda}{9}, \frac{-30 + 11\lambda}{9} \right)$ — A1 | |
| | **a** and **b** identified — M1 | |
| | Writing in required form; a correct equation — A1 | |

[FP3/P6 June 2005 Qn 3]

| | | |
|---|---|---|
| **37.** | (a) Det = –12 –2(2$k$ – 8) + 16 = 20 – 4$k$ (*) AG | M1A1 (2) |
| | (b) Cofactors $\begin{pmatrix} -4 & 8-2k & 4 \\ 8-2k & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$ [A1 each error] | M1A3 |
| | $\mathbf{A}^{-1} = \dfrac{1}{20-4k}\begin{pmatrix} -4 & 8-2k & 4 \\ 8-2k & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$ | M1A1√ (6) |
| | (c) Setting $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ | M1 |
| | $\lambda = -1$ | A1 (2) |
| | (d) Forming equations in $x$, $y$ and $z$: $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ | M1 |
| | $-5x + 2y + 4z = 0$, $2x + 2z = 8y$, $4x + 2y - 5z = 0$ | A1 |
| | Establishing ratio $x: y: z$ : [$x = 2y$, $x = z$] | M1 |
| | Eigenvector $(k)\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ | A1 (4) |
| | | **[14]** |

[FP3/P6 June 2005 Qn 7]

| | | |
|---|---|---|
| **38.** | $x^2 - 2x + 17 = (x-1)^2 + 16$ | B1 |
| | $I = \int_1^4 \frac{1}{\sqrt{(x-1)^2 + 16}} dx = [\text{arsinh} \frac{(x - 1)}{4}]$ or equiv. = arsinh ¾ | M1 A1√ |
| | [ M1 does not require limits; A1 f.t. on completing square, providing arsinh ] | |
| | Into ln form [ $\ln\left[\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right]$ ; = ln2 | M1A1 |
| | | **[5]** |
| | [ If straight to ln form : B1, $\ln\left[(x-1) + \sqrt{(x-1)^2 + 16}\right]$ M1 | |
| | Using limits correctly M1A1√ , ln2 A1 ] | |

[FP2/P5 January 2006 Qn 1]

| | | |
|---|---|---|
| **39.** | (a) Using $b^2 = a^2 (e^2 - 1)$ ; [ $4 = 16 (e^2 - 1)$ ] $e = \frac{\sqrt{5}}{2}$ or equiv. (1.12) | M1A1 **(2)** |
| | (b) Distance between foci = 2 a e [ $2 \times 4 \times \frac{\sqrt{5}}{2}$ ] ; = $4\sqrt{5}$ | M1A1√ **(2)** |
| | [ A1√ dependent on both Ms ] | |
| | (c) Ellipse, centred on origin | B1 |
| | Hyperbola, both branches | B1 |
| | Totally correct, touching, with correct intercepts | B1 |
| | | **(3)** |
| | | **[7]** |

[FP2/P5 January 2006 Qn 2]

| | | |
|---|---|---|
| **40.** | $\frac{dx}{dt} = 1 + \cos t, \qquad \frac{dy}{dt} = \sin t,$ (both) | B1 |
| | $s = \int \sqrt{((1+\cos t)^2 + (\sin t)^2)}\ dt$ ; $= \int \sqrt{2 + 2\cos t}\ dt$ | M1A1 |
| | Use of "half-angle formula" $[\int \sqrt{4\cos^2 t}\ dt]$; s = $\left[4 \sin\frac{t}{2}\right]_{(0)}^{\left(\frac{\pi}{2}\right)}$ | M1A1√ |
| | Using limits correctly and surd form; $= 2\sqrt{2}$ (allow $\frac{4}{\sqrt{2}}$ ) | M1A1 **[7]** |

[FP2/P5 January 2006 Qn 3]

| | | |
|---|---|---|
| **41.** | Using $\cosh x = \frac{e^x + e^{-x}}{2}$ and attempt to progress | M1 |
| | Correct intermediate step as far as $4\left(\frac{e^{3x} + 3e^x + 3e^{-x} + e^{-3x}}{8}\right) - \left[3\left(\frac{e^x + e^{-x}}{2}\right)\right]$ | A1 |
| | $= \frac{e^{3x} + e^{-3x}}{2} = \cosh 3x$ | A1 **(3)** |
| | (b) Using part (a) to reduce to $\cosh^2 x = [\ 2\ ]$ | M1 |
| | Correct method to form $\ln x$ or find $e^X$ or $e^{2x}$ | M1 |
| | $x = \ln(\sqrt{2}+1)$, $\ln(\sqrt{2}-1)$ or equivalent | A1 A1√ |
| | or ½ $\ln(3+2\sqrt{2})$, ½ $\ln(3-2\sqrt{2})$, ( after finding $e^{2x}$ = ……) | **(4)** |
| | | **[7]** |

[FP2/P5 January 2006 Qn 4]

| | | |
|---|---|---|
| **42.** | (a) $I_n = -\frac{2}{3}\left[x^n(4-x)^{\frac{3}{2}}\right]_0^4 + \frac{2}{3}n\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\,dx$ | M1A1 |
| | $= \frac{2}{3}n\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\,dx$ | A1√ |
| | $= \frac{2}{3}n\int_0^4 4x^{n-1}(4-x)^{\frac{1}{2}}\,dx - \frac{2}{3}n\int_0^4 x^n(4-x)^{\frac{1}{2}}\,dx$ | M1A1 |
| | $\Rightarrow I_n = \frac{8}{3}nI_{n-1} - \frac{2}{3}nI_n$ | |
| | [ $(2n+3)I_n = 8nI_{n-1}$ ] $\Rightarrow I_n = \frac{8n}{2n+3}I_{n-1}$ AG | A1* (6) |
| | (b) Relating $I_2$ to $I_0$ using result from (a) | M1 |
| | $I_2 = \frac{16}{7}\cdot\frac{8}{5}\quad I_0 = \frac{2048}{105}\quad\left(19\frac{53}{105}\right)$ | A1A1 (3) |
| | | **[9]** |

[FP2/P5 January 2006 Qn 7]

| | | |
|---|---|---|
| **43.** | (a) $\text{ar tanh}\,\frac{1}{\sqrt{2}} = \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)$ or equivalent | M1 |
| | $= \frac{1}{2}\ln\left[\frac{\sqrt{2}+1}{\sqrt{2}-1} \text{ x } \frac{\sqrt{2}+1}{\sqrt{2}+1}\right]$ or equivalent | M1 |
| | $= \frac{1}{2}\ln(\sqrt{2}+1)^2 = \ln(\sqrt{2}+1)$ AG | A1* (3) |
| | *Alternative Approach* | |
| | If using $y = \text{ar tanh}(\sin x) \Rightarrow \tanh y = \left(\frac{1}{\sqrt{2}}\right)$ [ or $\cosh^2 y = 2$ ] | |
| | and then use exponentials: | |
| | Progression as far as $e^y = \ldots\ldots$ *or* $e^{2y} = \ldots\ldots$ M1<br>Converting to ln form M1<br>Answer as given A1* | |
| | Note: $\frac{1}{2}\ln(3+2\sqrt{2})$ can earn M1M1 but for A1* there must be a convincing further step. | |
| | (b) $\frac{dy}{dx} = \frac{1}{1-\sin^2 x}.\cos x; \quad = \frac{\cos x}{\cos^2 x} = \sec x$ | M1A1 (2) |
| | Note: If $\tanh y = \sin x$ is differentiated M1 requires $\frac{dy}{dx} = f(x)$ | |
| | (c) Attempt at by parts and use of result in (b) | M1 |
| | $= -\cos x\,\text{ar tanh}(\sin x) + \int \cos x \sec x\,d\,dx$ | A1 |
| | | M1 |
| | $= -\cos x\,\text{ar tanh}(\sin x) + x$ | |
| | Using limits correctly : $= -\frac{1}{\sqrt{2}}\ln(1+\sqrt{2}) + \frac{\pi}{4}$ or exact equivalent | M1A1 (5) **[10 ]** |

[FP2/P5 January 2006 Qn 8]

| | | |
|---|---|---|
| **44.** | (*a*) $\begin{vmatrix} k-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$ | M1 |
| | Characteristic equation: $\lambda^2 - \lambda - 6 = 0$ | A1 |
| | Solving : $(\lambda - 3)(\lambda + 2) = 0 \Rightarrow \lambda = \ldots$ | M1 |
| | $\lambda = -2,\ \lambda = 3$ (both) | A1 (4) |
| | (*b*) Method for finding an eigenvector $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and | M1 |
| | $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ | A1√ |
| | Equations are: $y = \frac{1}{2}x$ and $y = -2x$. | A1 (3) |
| | **Alt:** $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} y \\ my \end{pmatrix} \Rightarrow 2m^2 + 3m - 2 = 0$ M1A1<br>[ $m$ = ½, –2] Correct equations A1 | **Total 9 marks** |

[*FP3/P6 January 2006 Qn 3]

| | | |
|---|---|---|
| **45.** | $$\mathbf{A} = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}$$ | |
| | (a) Det. $\mathbf{A} = -k^2 + 9k - 18$ | M1A1 |
| | Setting to zero and solving for $k$ $[(k-6)(k-3)=0]$ | M1 |
| | $\Rightarrow k = 3,\ k = 6$ | A1 (4) |
| | (b) Cofactors $\begin{pmatrix} -k & 9k & 9 \\ -2 & 18 & 9-k \\ k-2 & -k^2 & -k \end{pmatrix}$ | B3 |
| | [B1 for each row (or column)] | |
| | $\mathbf{A}^{-1} = \frac{1}{\det}\begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & 9-k & -k \end{pmatrix}$ | M1A1√ (5) |
| | [A1 f.t. is on determinant or cofactors ] | **Total 9 marks** |

[FP3/P6 January 2006 Qn 4]

| | | |
|---|---|---|
| **46.** | (a) $\vec{RQ} = \begin{pmatrix} 1 \\ -1 \\ -1-c \end{pmatrix}$, $RP = \begin{pmatrix} -4 \\ 3 \\ -2-c \end{pmatrix}$ (both) | B1 |
| | $\vec{RP} \times \vec{RQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -2-c \\ 1 & -1 & -1-c \end{vmatrix}$ | |
| | $= (-5-4c)\,\mathbf{i} - (6+5c)\,\mathbf{j} + \mathbf{k}$ | M1A1√ (3) |
| | (b) $c = -2$ <br> $d = -6 - 5c = 4$ AG | A1√ <br> A1*(cso) (2) |
| | (c) $\mathbf{r}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = p$ | M1 |
| | Substituting point in plane to give $p$, $\mathbf{r}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 7$. | M1A1 (3) |
| | (d) Equation of normal to plane through S : $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ | B1 |
| | Meets plane where $\begin{pmatrix} 1+3t \\ 5+4t \\ 10+t \end{pmatrix}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 7 \quad \Rightarrow t = -1$ | M1A1√ |

[FP3/P6 January 2006 Qn 7]

| | | |
|---|---|---|
| **47.** | $5\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$ | B1 |
| | $3e^{2x} - 22e^x + 7 = 0$ M: Simplify to form <u>quadratic</u> in $e^x$ | M1 A1 |
| | $(3e^x - 1)(e^x - 7) = 0$ $e^x = \frac{1}{3}$, $e^x = 7$ M: Solve 3 term quadratic. | M1 A1 |
| | $x = \ln\frac{1}{3}\ (\text{or} - \ln 3)$ $x = \ln 7$ | A1 (6) |
| | | **6 Marks** |

[FP2 June 2006 Qn 1]

| | | |
|---|---|---|
| **48.** (*a*) | Using $b^2 = a^2(1 - e^2)$ or equiv. to find *e* or *ae*: (*a* = 2 and *b* = 1) | |
| | $e = \frac{\sqrt{3}}{2}$ | M1 A1 |
| | Using $y^2 = 4(ae)x$ $y^2 = 4\sqrt{3}x$ (M requires <u>values</u> for *a* and *e*) | M1 A1 (4) |
| (*b*) | $x = -\sqrt{3}$ | B1ft (1) |
| | | **5 Marks** |

[FP2 June 2006 Qn 2]

| | | | |
|---|---|---|---|
| **49.** | (a) | $\frac{dy}{dx} = 4\operatorname{sech}^2 4x - 1$ | B1 |
| | | Put $\frac{dy}{dx} = 0$ $\left(\cosh^2 4x = 4 \quad \cosh 4x = 2\right)$ | M1 |
| | | $4x = \ln(2 \pm \sqrt{3})$ or $8x = \ln(7 \pm 4\sqrt{3})$ or $e^{4x} = 2 \pm \sqrt{3}$ or $e^{4x} = 7 \pm 4\sqrt{3}$ (± or +) | A1 |
| | | $x = \frac{1}{4}\ln(2 + \sqrt{3})$ or $x = \frac{1}{8}\ln(7 + 4\sqrt{3})$ (or equiv.) | A1 (4) |
| | (b) | $y = -\frac{1}{4}\ln(2 + \sqrt{3}) + \tanh(....)$ (Substitute for $x$) | M1 |
| | | $\operatorname{sech} 4x = \frac{1}{2} = \sqrt{1 - \tanh^2 4x}$, $\tanh 4x = \frac{\sqrt{3}}{2}$ | M1 |
| | | $y = \frac{\sqrt{3}}{2} - \frac{1}{4}\ln(2 + \sqrt{3}) = \frac{1}{4}\left\{2\sqrt{3} - \ln\left(2 + \sqrt{3}\right)\right\}$ (*) | A1 (3) |
| | | | **7 Marks** |
| | | (a) 'Second solution', if seen, must be rejected to score the final mark.<br>(b) 2nd M requires an expression in terms of √3 without hyperbolics, exponentials and logarithms. | |

[FP2 June 2006 Qn 5]

| | | | |
|---|---|---|---|
| **50.** | (a) | $\frac{dx}{dt} = 1 - \frac{1}{t} \qquad \frac{dy}{dt} = 2t^{-\frac{1}{2}}$ | B1 B1 |
| | | $\sqrt{\left(1-\frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2}, \qquad = \sqrt{1+\frac{2}{t}+\frac{1}{t^2}} = 1+\frac{1}{t}$ or $\frac{t+1}{t}$ | M1, A1 |
| | | Length $= \int_1^4 \left(1+\frac{1}{t}\right)dt = \left[t + \ln t\right]_1^4 = (4+\ln 4) - 1 = 3 + \ln 4 \quad (*)$ | M1 M1 A1 (7) |
| | (b) | Surface area = | |
| | | $2\pi\int_1^4 4\sqrt{t}\sqrt{\left(1-\frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2}\,dt \qquad \left(= 8\pi\int_1^4 \left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right)dt\right)$ | M1 |
| | | $= (8\pi)\left[\frac{2t^{\frac{3}{2}}}{3} + 2t^{\frac{1}{2}}\right]_1^4 = (8\pi)\left\{\left(\frac{16}{3}+4\right) - \left(\frac{2}{3}+2\right)\right\} = \frac{160\pi}{3} \qquad \left(53\frac{1}{3}\pi\right)$ | M1 M1 A1 (4) |
| | | | **(11 marks)** |

[FP2 June 2006 Qn 6]

| | | |
|---|---|---|
| **51.** | $\int x^2 \operatorname{arsinh} x\,dx = \frac{x^3}{3}\operatorname{arsinh} x - \int \frac{x^3}{3\sqrt{x^2+1}}dx$ | M1 A1 A1 |
| | $\left[\frac{x^3}{3}\operatorname{arsinh} x\right]_0^3 = 9\operatorname{arsinh}3 \qquad \left(\text{or } 9\ln\left(3+\sqrt{10}\right)\right)$ | B1 |
| | Let $u = x^2 + 1 \qquad \frac{du}{dx} = 2x \qquad \left[u^2 = x^2 + 1 \qquad 2u\frac{du}{dx} = 2x\right]$ | M1 |
| | $\frac{1}{3}\int \frac{x^3}{u^{\frac{1}{2}}.2x}du = \frac{1}{6}\int \frac{u-1}{u^{\frac{1}{2}}}du = \frac{1}{6}\int\left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du \qquad \left[\frac{1}{3}\int\left(u^2-1\right)du\right]$ | M1 |
| | $= \frac{1}{6}\left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}}\right] \qquad \left[= \frac{1}{3}\left[\frac{u^3}{3} - u\right]\right]$ | M1 |
| | When $x = 0$, $u = 1$ and when $x = 3$, $u = 10 \qquad \left[\ldots\ldots u = \sqrt{10}\right]$ | |
| | $\frac{1}{6}\left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}}\right]_1^{10} = \frac{1}{6}\left\{\left(\frac{20\sqrt{10}}{3} - 2\sqrt{10}\right) - \left(\frac{2}{3} - 2\right)\right\}$ | M1 A1 |
| | Area $= 9\operatorname{arsinh}3 - \frac{1}{6}\left(\frac{14\sqrt{10}}{3} + \frac{4}{3}\right) = 9\ln\left(3+\sqrt{10}\right) - \frac{1}{9}\left(7\sqrt{10}+2\right) \qquad (*)$ | A1cso (10) |

| | | |
|---|---|---|
| | Dependent M marks:<br>M: Choose an appropriate substitution & find $\frac{\mathrm{d}u}{\mathrm{d}x}$ or 'Set up' integration by parts.<br>M: Get all in terms of '$u$' or Use integration by parts.<br>M: Sound integration.<br>M: Substitute both limits (for the correct variable) and subtract. | |

| | | |
|---|---|---|
| **51.** | Alternative solution: | |
| | Let $x = \sinh\theta$ $\quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cosh\theta$ | M1 |
| | $\int x^2 \operatorname{ar}\sinh x \, \mathrm{d}x = \int \theta \sinh^2\theta \cosh\theta \, \mathrm{d}\theta$ | M1 |
| | $= \left[\frac{\theta \sinh^3\theta}{3}\right] - \int \frac{1}{3}\sinh\theta(\cosh^2\theta - 1)\mathrm{d}\theta$ | M1 A1 A1 |
| | $\left[\frac{\theta\sinh^3\theta}{3}\right]_0^{\operatorname{arsinh}3} = 9\operatorname{arsinh}3$ | B1 |
| | $\int \frac{1}{3}\sinh\theta(\cosh^2\theta - 1)\mathrm{d}\theta = \frac{1}{3}\left[\frac{\cosh^3\theta}{3} - \cosh\theta\right]$ | M1 |
| | $\frac{1}{3}\left[\frac{\cosh^3\theta}{3} - \cosh\theta\right]_0^{\operatorname{arsinh}3} = \frac{1}{3}\left\{\left(\frac{10\sqrt{10}}{3} - \sqrt{10}\right) - \left(\frac{1}{3} - 1\right)\right\}$ | M1 A1 |
| | Area $= 9\operatorname{arsinh}3 - \frac{1}{3}\left(\frac{7\sqrt{10}}{3} + \frac{2}{3}\right) = 9\ln\left(3+\sqrt{10}\right) - \frac{1}{9}\left(7\sqrt{10} + 2\right)$ (*) | A1cso<br>(10) |
| | | **10 Marks** |

| | | |
|---|---|---|
| **51.** | <u>A few alternatives for:</u> $\int \frac{x^3}{\sqrt{x^2+1}}\mathrm{d}x$ . | |
| (i) | Let $u = x^2$ $\quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ | |
| | $\int \frac{u^{\frac{3}{2}}}{\sqrt{1+u}} \cdot \frac{1}{2u^{\frac{1}{2}}}\mathrm{d}u = \frac{1}{2}\int \frac{u}{\sqrt{1+u}}\mathrm{d}u$ | |
| | No marks yet… needs another substitution, or parts, or perhaps… | |
| | $\frac{u}{\sqrt{1+u}} = \sqrt{1+u} - \frac{1}{\sqrt{1+u}}$ | M1 |
| | $\frac{1}{2}\int \sqrt{1+u}\,\mathrm{d}u - \frac{1}{2}\int \frac{1}{\sqrt{1+u}}\mathrm{d}u$ | M1 |
| | $\frac{1}{3}(1+u)^{\frac{3}{2}} - (1+u)^{\frac{1}{2}}$ | M1 |
| | Limits (0 to 9) | M1 |
| (ii) | Let $x = \sinh\theta$ $\quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cosh\theta$ | M1 |
| | $\int \frac{\sinh^3\theta}{\cosh\theta} \cdot \cosh\theta\,\mathrm{d}\theta = \int \sinh\theta(\cosh^2\theta - 1)\mathrm{d}\theta$ | M1 |
| | Then, as in the alternative solution, | |
| | $\int \frac{1}{3}\sinh\theta(\cosh^2\theta - 1)\mathrm{d}\theta = \frac{1}{3}\left[\frac{\cosh^3\theta}{3} - \cosh\theta\right]$ | M1 |
| | Limits (0 to arsinh3) | M1 |

| | | |
|---|---|---|
| **51.** (iii) | Let $u = \tan\theta \qquad \frac{\mathrm{d}u}{\mathrm{d}\theta} = \sec^2\theta$ | M1 |
| | $\int \frac{\tan^3\theta}{\sec\theta}\cdot\sec^2\theta\,\mathrm{d}\theta = \int \tan\theta\sec\theta\left(\sec^2\theta - 1\right)\mathrm{d}\theta$ | M1 |
| | $= \int \sec^2\theta(\sec\theta\tan\theta)\mathrm{d}\theta - \int(\sec\theta\tan\theta)\mathrm{d}\theta = \frac{\sec^3\theta}{3} - \sec\theta$ | M1 |
| | Limits $\left(\sec\theta = 1 \text{ to } \sec\theta = \sqrt{10}\right)$ | M1 |
| (iv) | (By parts… must be the 'right way round', not integrating $x^2$) | |
| | $u = x^2,\ \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x}{\sqrt{1+x^2}},\ v = \sqrt{1+x^2}$ | M1 |
| | $x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2}\,\mathrm{d}x$ | M1 |
| | $x^2\sqrt{1+x^2} - \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$ | M1 |
| | Limits | M1 |
| (v) | (By parts) | |
| | $u = x^3,\ \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{\sqrt{1+x^2}},\ v = \operatorname{arsinh} x$ No progress | M0 |
| (vi) | $\frac{x^3}{\sqrt{1+x^2}} = \frac{x(x^2+1)-x}{\sqrt{1+x^2}} = \frac{x(x^2+1)}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}$ | M1 |
| | $\int x\sqrt{1+x^2}\,\mathrm{d}x - \int \frac{x}{\sqrt{1+x^2}}\mathrm{d}x$ | M1 |
| | $= \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \left(1+x^2\right)^{\frac{1}{2}}$ | M1 |
| | Limits | M1 |

[FP2 June 2006 Qn7]

| | | | |
|---|---|---|---|
| **52.** | (a) | $\int x^n \cosh x\,dx = x^n \sinh x - \int nx^{n-1} \sinh x\,dx$ | M1 A1 |
| | | $= x^n \sinh x - nx^{n-1}\cosh x + n(n-1)\int x^{n-2}\cosh x\,dx$ | M1 |
| | | $I_n = x^n \sinh x - nx^{n-1}\cosh x + n(n-1)I_{n-2}$ (*) | A1 (4) |
| | (b) | $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12I_2$ | M1 |
| | | $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12\left(x^2 \sinh x - 2x\cosh x + 2I_0\right)$ | M1 |
| | | (This M may also be scored by finding $I_2$ by integration.) | |
| | | $I_0 = \int \cosh x\,dx = \sinh x + k$ | B1 |
| | | $I_4 = \left(x^4 + 12x^2 + 24\right)\sinh x,\ +\left(-4x^3 - 24x\right)\cosh x \quad (+C)$ | A1, A1 (5) |
| | (c) | $\left[\left(x^4 + 12x^2 + 24\right)\sinh x + \left(-4x^3 - 24x\right)\cosh x\right]_0^1$ | |
| | | = 37sinh 1 – 28cosh 1 M: $x$ = 1 substituted throughout (at some stage) | M1 |
| | | $= 37\left(\frac{e - e^{-1}}{2}\right) - 28\left(\frac{e + e^{-1}}{2}\right)$ | M1 |
| | | M: Use of exp. Definitions (can be in terms of $x$) | |
| | | $= \frac{1}{2}\left(9e - 65e^{-1}\right)$ | A1 (3) |
| | | | **12 Marks** |
| | | **(b) Integration constant missing <u>throughout</u> loses the B mark** | |

[FP2 June 2006 Qn 8]

| | | |
|---|---|---|
| **53.**(a) | $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1 \qquad b^2x^2+a^2(mx+c)^2=a^2b^2$ | M1 |
| | $(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$ (*) | A1 (2) |
| (b) | $(2a^2mc)^2=4(b^2+a^2m^2)a^2(c^2-b^2)$ | M1 |
| | $4a^4m^2c^2=4a^2(b^2c^2-b^4+a^2m^2c^2-a^2m^2b^2) \qquad c^2=b^2+a^2m^2$ (*) | A1 (2) |
| (c) | Find height and base of triangle (perhaps in terms of $c$). | M1 |
| | $OB=c \quad \left(=\sqrt{b^2+a^2m^2}\right)$ and $AO=\dfrac{c}{m} \quad \left(=\dfrac{\sqrt{b^2+a^2m^2}}{m}\right)$ | A1 |
| | Area of triangle $OAB=\dfrac{c^2}{2m}=\dfrac{b^2+a^2m^2}{2m}$ M: Find area and subs. for $c$. | M1 A1 (4) |
| (d) | $\Delta=\dfrac{b^2+a^2m^2}{2m}=\dfrac{b^2}{2}m^{-1}+\dfrac{a^2}{2}m$ <br> $\dfrac{d\Delta}{dm}=-\dfrac{b^2}{2}m^{-2}+\dfrac{a^2}{2}=0 \qquad \dfrac{b^2}{m^2}=a^2 \qquad m=\dfrac{b}{a}$ | M1 A1 |
| | $\Delta=\left(\dfrac{b^2}{2}\right)\left(\dfrac{a}{b}\right)+\left(\dfrac{a^2}{2}\right)\left(\dfrac{b}{a}\right)=ab$ (*) | A1 (3) |
| (e) | Root of quadratic: $x=\dfrac{-a^2mc}{b^2+a^2m^2}$ (Should be <u>correct</u> if quoted directly) | M1 |
| | Using $m=\dfrac{b}{a}$ and $c=\sqrt{b^2+a^2m^2}$: $x=-\dfrac{a}{\sqrt{2}}$ | M1 A1 (3) |
| | (The 2nd M is dependent on using the quadratic equation). | **14 Marks** |
| (d) | <u>Alternative:</u> $b^2+a^2m^2\geq 2bam$ (since $(b-am)^2\geq 0$) | [M1] |
| | $\dfrac{b^2+a^2m^2}{2m}\geq ab$ [A1] | |
| | Conclusion [A1] | |
| (e) | <u>Alternative:</u> | |
| | Begin with full eqn. $(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$. | |
| | In the eqn., use conditions $m=\dfrac{b}{a}$ and $c=\sqrt{b^2+a^2m^2}$ $(=b\sqrt{2})$ | [M1] |
| | Simplify and solve eqn., e.g. $2x^2+2a\sqrt{2}x+a^2=0 \quad x=-\dfrac{a}{\sqrt{2}}$ | |

[FP2 June 2006 Qn 9]

| | | |
|---|---|---|
| **54.** | $$\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1+3) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{A}$$ ( Hence true for $n$ =1) | B1 |
| | $$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A} = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$ | M1 |
| | $$= \begin{pmatrix} 1 & k+1 & 2+k+\frac{1}{2}(k^2+3k) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$ | |
| | $$2+k+\tfrac{1}{2}(k^2+3k) = \tfrac{1}{2}(k^2+5k+4) = \tfrac{1}{2}(k^2+2k+1+3k+3)$$ | M1 Dep |
| | $$= \tfrac{1}{2}\left((k+1)^2+3(k+1)\right)$$ | A1 |
| | ( Hence, if result is true for $n = k$, then it is true for $n = k + 1$). | |
| | By Mathematical Induction, above implies true for all positive integers. cso | A1 cso **(5)** |
| | | **[5 marks]** |

[FP3 June 2006 Qn 1]

| | | | |
|---|---|---|---|
| **55.** | (a) | $(4-\lambda)(1-\lambda)+2=0$ | M1 |
| | | $\lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0$ | M1 |
| | | $\lambda_1 = 2, \lambda_2 = 3$ both | A1 **(3)** |
| | (b) | $M^{-1} = \frac{1}{\underline{6}}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ | <u>B1</u> B1 **(2)** |
| | (c) | $\begin{vmatrix} \frac{1}{6}-\frac{1}{2} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3}-\frac{1}{2} \end{vmatrix} = -\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{6}=0$ M1 for either value | M1 A1 |
| | | (hence $\frac{1}{2}$ is an eigenvalue of $\mathbf{M}^{-1}$) | |

| | | |
|---|---|---|
| | $\begin{vmatrix} \frac{1}{6}-\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3}-\frac{1}{3} \end{vmatrix} = -\frac{1}{6}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{6}=0$ | A1 |
| | $\left(\text{hence } \frac{1}{3} \text{ is an eigenvalue of } \mathbf{M}^{-1}\right)$ | **(3)** |
| | (d) Using eigenvalues | |
| | $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}.\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$ | |
| | $4x-2y=2x \quad \Rightarrow \quad y=x$ | M1 A1 |
| | $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}.\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$ | |
| | $4x-2y=3x \quad \Rightarrow \quad y=\frac{1}{2}x$ | M1 A1 **(4)** |
| | | **[12]** |
| | *Alternative to* (c), using characteristic polynomial of $\mathbf{M}^{-1}$ | |
| | $\left(\frac{1}{6}-\lambda\right)\left(\frac{2}{3}-\lambda\right)+\frac{1}{3}\times\frac{1}{6}=0$ | M1 |
| | Leading to $6\lambda^2-5\lambda+1=(3\lambda-1)(2\lambda-1)=0 \Rightarrow \lambda=\frac{1}{2},\frac{1}{3}$ | A1, A1 **(3)** |
| | *Alternative to* (d) $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}.\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x' \\ mx' \end{pmatrix}$ | |
| | $4x-2mx=x', \quad x+mx=mx'$ both | M1 |
| | $\dfrac{1+m}{4-2m}=m$ | A1 |
| | Leading to $2m^2-3m+1=(2m-1)(m-1)=0 \Rightarrow m=\frac{1}{2},1$ | M1 |
| | $y=\frac{1}{2}x, \ y=x$ both | A1 **(4)** |

[FP3 June 2006 Qn 5]

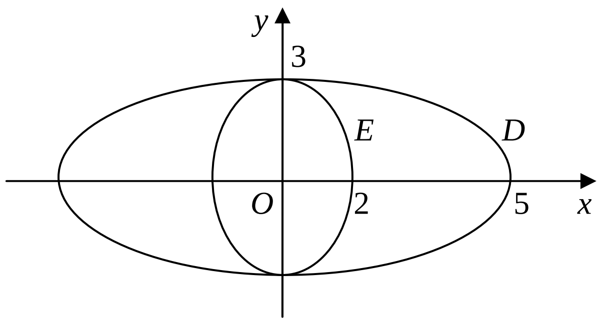
| | | |
|---|---|---|
| **56.** | (a) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -3\\ 4 & -5 & -1\end{vmatrix}$ | M1 |
| | $=-15\mathbf{i}-10\mathbf{j}-10\mathbf{k}$ | A1+A1+A1 **(4)** |
| | Allow M1 A1 for negative of above | |
| | (b) $\mathbf{r}.(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}).(\mathbf{i}+3\mathbf{j}-\mathbf{k})$ or equivalent | M1 |
| | $\mathbf{r}.(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=7$ or multiple | A1 **(2)** |
| | (c) Let $x=\lambda,\quad z=3-\lambda$, | |
| | then $2y=7-3\lambda-2(3-\lambda)\Rightarrow y=\frac{1}{2}-\frac{1}{2}\lambda$ | |
| | *x*, *y* and *z* in terms of a single parameter | M1 |
| | The direction of *l* is any multiple of $(2\mathbf{i}-\mathbf{j}-2\mathbf{k})$ | M1 |
| | $\left(\mathbf{r}-\left(\frac{1}{2}\mathbf{j}+3\mathbf{k}\right)\right)\times(2\mathbf{i}-\mathbf{j}-2\mathbf{k})=\mathbf{0}$ or equivalent | M1 A1 **(4)** |
| | Possible equivalents are $(\mathbf{r}-(\mathbf{i}+2\mathbf{k}))\times(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=\mathbf{0}$ | |
| | and $(\mathbf{r}-(3\mathbf{i}-\mathbf{j}))\times\left(-\mathbf{i}+\frac{1}{2}\mathbf{j}+\mathbf{k}\right)=\mathbf{0}$ | |
| | The general form is | |
| | $\{\mathbf{r}-[\mathbf{i}+2\mathbf{k}+c_1(2\mathbf{i}-\mathbf{j}-2\mathbf{k})]\}\times c_2(2\mathbf{i}-\mathbf{j}-2\mathbf{k})=\mathbf{0}$ | |
| | (d) $\left(\lambda\mathbf{i}+\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\mathbf{j}+(3-\lambda)\mathbf{k}\right).(2\mathbf{i}-\mathbf{j}-2\mathbf{k})=0$ | M1 |
| | $2\lambda-\frac{1}{2}+\frac{1}{2}\lambda-6+2\lambda=0$ | |
| | Leading to $\lambda=\frac{13}{9}$ | M1 A1 |
| | $P:\left(\frac{13}{9},-\frac{2}{9},\frac{14}{9}\right)$ | A1 **(4)** |
| | | **[14]** |
| | *Alternative to* (d) | |

| | | |
|---|---|---|
| | $OP^2 = \lambda^2 + \left(\tfrac{1}{2} - \tfrac{1}{2}\lambda\right)^2 + (3-\lambda)^2 \quad \left(= \tfrac{1}{4}\left(9\lambda^2 - 26\lambda + 37\right)\right)$ | M1 |
| | $\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\left(OP^2\right) = 0 \quad \Rightarrow \quad \lambda = \tfrac{13}{9}$ | M1 A1 |
| | $P : \left(\tfrac{13}{9}, -\tfrac{2}{9}, \tfrac{14}{9}\right)$ | A1 **(4)** |

[FP3 June 2006 Qn 7]

| | | |
|---|---|---|
| **57.** | $x^2 + 4x - 5 = (x+2)^2 - 9$ | B1 |
| | $\displaystyle\int \frac{1}{\sqrt{\left((x+2)^2 - 9\right)}}\,\mathrm{d}x = \operatorname{arcosh}\frac{x+2}{3}$ | M1 A1ft |
| | ft their completing the square, requires arcosh | |
| | $\left[\operatorname{arcosh}\dfrac{x+2}{3}\right]_1^3 = \operatorname{arcosh}\dfrac{5}{3}\ \left(-\operatorname{arcosh} 1\right)$ | |
| | $= \ln\left(\dfrac{5}{3} + \sqrt{\dfrac{25}{9} - 1}\right) = \ln\left(\dfrac{5}{3} + \dfrac{4}{3}\right) = \ln 3$ | M1 A1 **(5)** |
| | | **[5]** |
| | *Alternative* | |
| | $x^2 + 4x - 5 = (x+2)^2 - 9$ | B1 |
| | Let $x + 2 = 3\sec\theta$, $\quad \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sec\theta\tan\theta$ | |
| | $\displaystyle\int \frac{1}{\sqrt{\left((x+2)^2 - 9\right)}}\,\mathrm{d}x = \int \frac{3\sec\theta\tan\theta}{\sqrt{\left(9\sec^2\theta - 9\right)}}\,\mathrm{d}\theta$ | M1 |
| | $= \displaystyle\int \sec\theta\,\mathrm{d}\theta$ | A1ft |
| | $\left[\ln\left(\sec\theta + \tan\theta\right)\right]_{\operatorname{arcsec}1}^{\operatorname{arcsec}\frac{5}{3}} = \ln\left(\dfrac{5}{3} + \dfrac{4}{3}\right) = \ln 3$ | M1 A1 **(5)** |

[FP2 June 2007 Qn 1]

| | | | |
|---|---|---|---|
| **58.** | (a)   | | |
| | One ellipse centred at $O$ | B1 | |
| | Another ellipse, centred at $O$, touching on $y$-axis | B1 | |
| | Intersections: At least 2, 5, and 3 shown correctly | B1 | **(3)** |
| | (b) Using $b^2 = a^2\left(1-e^2\right)$, or equivalent, to find $e$ or $ae$ for $D$ or $E$. | M1 | |
| | For $S$: $a=5$ and $b=3$, $e=\frac{4}{5}$, $ae=4$ ignore sign with $ae$ | A1 | |
| | For $T$: $a'=3$ and $b'=2$, $e'=\frac{\sqrt{5}}{3}$, $a'e'=\sqrt{5}$ ignore sign with $a'e'$ | A1 | |
| | $ST=\sqrt{(16+5)}=\sqrt{21}$ | M1 A1 | **(5)** <br> **[8]** |

[FP2 June 2007 Qn 2]

| 59. | | |
|---|---|---|
| | $\frac{dy}{dx} = \frac{1}{4}\left(4x - \frac{1}{x}\right)$ | B1 |
| | $\int\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2} dx = \int\left(1+\left(x-\frac{1}{4x}\right)^2\right)^{1/2} dx$ | M1 |
| | $= \int\left(1+x^2+\frac{1}{16x^2}-\frac{1}{2}\right)^{1/2} dx = \int\left(\left(x+\frac{1}{4x}\right)^2\right)^{1/2} dx = \int\left(x+\frac{1}{4x}\right)dx$ | M1 A1 |
| | $= \frac{x^2}{2}+\frac{\ln x}{4}$ | A1 |
| | $\left[\frac{x^2}{2}+\frac{\ln x}{4}\right]_{0.5}^{2} = 2+\frac{\ln 2}{4}-\frac{1}{8}-\frac{\ln 0.5}{4} = \frac{15}{8}+\frac{1}{2}\ln 2$ | M1 A1 **(7)** |
| | $\left(a=\frac{15}{8},\ b=\frac{1}{2}\right)$ | **(7 marks)** |

[FP2 June 2007 Qn 3]

| 60. | | |
|---|---|---|
| | (a) | |
| | $\cosh A\cosh B - \sinh A\sinh B = \left(\frac{e^A+e^{-A}}{2}\right)\left(\frac{e^B+e^{-B}}{2}\right)-\left(\frac{e^A-e^{-A}}{2}\right)\left(\frac{e^B-e^{-B}}{2}\right)$ | M1 |
| | $= \frac{1}{4}\left(e^{A+B}+e^{-A+B}+e^{A-B}+e^{-A-B}-e^{A+B}+e^{-A+B}+e^{A-B}-e^{-A-B}\right)$ | |
| | $= \frac{1}{4}\left(2e^{-A+B}+2e^{A-B}\right) = \frac{e^{A-B}+e^{-(A-B)}}{2} = \cosh(A-B)$ ∗ cso | M1 A1 **(3)** |
| | (b) $\cosh x\cosh 1 - \sinh x\sinh 1 = \sinh x$ | M1 |
| | $\cosh x\cosh 1 = \sinh x(1+\sinh 1) \Rightarrow \tanh x = \frac{\cosh 1}{1+\sinh 1}$ | M1 |
| | $\tanh x = \frac{\frac{e+e^{-1}}{2}}{1+\frac{e-e^{-1}}{2}} = \frac{e+e^{-1}}{2+e-e^{-1}} = \frac{e^2+1}{e^2+2e-1}$ ∗ cso | M1 A1 **(4)** |
| | | **[7]** |
| | *Alternative for* (b) | |
| | $\frac{e^{x-1}+e^{-(x-1)}}{2} = \frac{e^x-e^{-x}}{2}$ | M1 |

| | | |
|---|---|---|
| | Leading to $\quad e^{2x} = \dfrac{e^2 + e}{e - 1}$ | M1 |
| | $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{e^2 + e - (e-1)}{e^2 + e + (e-1)} = \dfrac{e^2 + 1}{e^2 + 2e - 1}$ $\ast$ cso | M1 A1 **(4)** |

[FP2 June 2007 Qn 4]

| | | |
|---|---|---|
| **61.** | (a) $I_n = -\dfrac{3}{4}\left[ x^n (8-x)^{4/3} \right]_0^8 + \dfrac{3}{4}\int nx^{n-1}(8-x)^{4/3}\,dx$ | M1 A1 |
| | $= \dfrac{3}{4}\int nx^{n-1}(8-x)^{4/3}\,dx$ ft numeric constants only | A1ft |
| | $\int nx^{n-1}(8-x)(8-x)^{1/3}\,dx = \int nx^{n-1}8(8-x)^{1/3}\,dx - \int nx^{n-1}x(8-x)^{1/3}\,dx$ | M1 A1 |
| | $I_n = 6nI_{n-1} - \dfrac{3}{4}nI_n \Rightarrow I_n = \dfrac{24n}{3n+4}I_{n-1}$ $\ast$ cso | A1 **(6)** |
| | (b) $I_0 = \int_0^8 (8-x)^{1/3}\,dx = \left[ -\dfrac{3}{4}(8-x)^{4/3} \right]_0^8 = \dfrac{3}{4}\times 8^{4/3} = 12$ | M1 A1 |
| | $I = \int_0^8 x(x+5)(8-x)^{1/3}\,dx = I_2 + 5I_1$ | M1 |
| | $I_1 = \dfrac{24}{7}I_0, \quad I_2 = \dfrac{48}{10}I_1 = \dfrac{48}{10}\times\dfrac{24}{7}I_0 \left( = \dfrac{576}{35}I_0 \right)$ | M1 A1 |
| | (The previous line can be implied by $I = I_2 + 5I_1 = \dfrac{168}{5}I_0$) | |
| | $I = \left( \dfrac{576}{35} + 5\times\dfrac{24}{7} \right)\times 12 = \dfrac{2016}{5} \; (= 403.2)$ | A1 **(6)** |
| | | **(12 marks)** |

[FP2 June 2007 Qn 6]

| | | | |
|---|---|---|---|
| **62.** | (a) $\dfrac{d}{dx}\left(\text{arsinh } x^{1/2}\right) = \dfrac{1}{\sqrt{(1+x)}} \times \dfrac{1}{2} x^{-1/2} \quad \left( = \dfrac{1}{2\sqrt{x}\sqrt{(1+x)}} \right)$ | M1 A1 | |
| | At $x = 4$, $\dfrac{dy}{dx} = \dfrac{1}{4\sqrt{5}}$ accept equivalents | A1 | **(3)** |
| | (b) $x = \sinh^2\theta$, $\dfrac{dx}{d\theta} = 2\sinh\theta\cosh\theta$ | | |
| | $\int \text{arsinh}\sqrt{x}\,dx = \int \theta \times 2\sinh\theta\cosh\theta\,d\theta$ | M1 A1 | |
| | $= \int \theta\sinh 2\theta\,d\theta = \dfrac{\theta\cosh 2\theta}{2} - \int \dfrac{\cosh 2\theta}{2}\,d\theta$ | M1 A1 + A1 | |
| | $= \ \dots \ - \dfrac{\sinh 2\theta}{4}$ | M1 | |
| | $\left[\dfrac{\theta\cosh 2\theta}{2} - \dfrac{\sinh 2\theta}{4}\right]_0^{\text{arsinh } 2} = \ \dots$ attempt at substitution | M1 | |
| | $= \left[\dfrac{\theta\left(1+2\sinh^2\theta\right)}{2} - \dfrac{2\sinh\theta\cosh\theta}{4}\right] = \dfrac{1}{2}\text{arsinh } 2 \times (1+8) - \dfrac{4\sqrt{5}}{4}$ | M1 A1 | |
| | $= \dfrac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$ | A1 | **(10)** |
| | | | **[13]** |

| | | | |
|---|---|---|---|
| | *Alternative for* (a) | | |
| | $x = \sinh^2 y$, $2\sinh y\cosh y\dfrac{dy}{dx} = 1$ | M1 | |
| | $\dfrac{dy}{dx} = \dfrac{1}{2\sinh y\cosh y} = \dfrac{1}{2\sinh y\sqrt{\left(\sinh^2 y + 1\right)}} \left( = \dfrac{1}{2\sqrt{x}\sqrt{(1+x)}} \right)$ | A1 | |
| | At $x = 4$, $\dfrac{dy}{dx} = \dfrac{1}{4\sqrt{5}}$ accept equivalents | A1 | **(3)** |
| | *An alternative for* (b) *is given on the next page* | | |

| | | | |
|---|---|---|---|
| **62.** | *Alternative for* (b) | | |
| | $\int 1\times \text{arsinh}\sqrt{x}\,dx = x\,\text{arsinh}\sqrt{x} - \int x\times\frac{1}{2\sqrt{x}\sqrt{(1+x)}}dx$ | M1 A1 + A1 | |
| | $= x\,\text{arsinh}\sqrt{x} - \int \frac{\sqrt{x}}{2\sqrt{(1+x)}}dx$ | | |
| | Let $x = \sinh^2\theta$, $\frac{dx}{d\theta} = 2\sinh\theta\cosh\theta$ | | |
| | $\int \frac{\sqrt{x}}{\sqrt{(1+x)}}dx = \int \frac{\sinh\theta}{\cosh\theta}\times 2\sinh\theta\cosh\theta\,d\theta$ | M1 A1 | |
| | $= 2\int \sinh^2\theta\,d\theta = 2\int \frac{\cosh 2\theta - 1}{2}d\theta, \ = \frac{\sinh 2\theta}{2} - \theta$ | M1, M1 | |
| | $\left[\frac{\sinh 2\theta}{2} - \theta\right]_0^{\text{arsinh}\,2} = \left[\frac{2\sinh\theta\cosh\theta}{2} - \theta\right]_0^{\text{arsinh}\,2} = \frac{2\times 2\times\sqrt{5}}{2} - \text{arsinh}\,2$ | M1 A1 | |
| | $\int_0^4 \text{arsinh}\sqrt{x}\,dx = 4\,\text{arsinh}\,2 - \frac{1}{2}\left(\frac{2\times 2\times\sqrt{5}}{2} - \text{arsinh}\,2\right) = \frac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$ | A1 | **(10)** |
| | *The last 7 marks of the alternative solution can be gained as follows* | | |
| | Let $x = \tan^2\theta$, $\frac{dx}{d\theta} = 2\tan\theta\sec^2\theta$ | | |
| | $\int \frac{\sqrt{x}}{\sqrt{(1+x)}}dx = \int \frac{\tan\theta}{\sec\theta}\times 2\tan\theta\sec^2\theta\,d\theta$ dependent on first M1 | M1 A1 | |
| | $= \int 2\sec\theta\tan^2\theta\,d\theta$ | | |
| | $\int(\sec\theta\tan\theta)\tan\theta\,d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta$ | M1 | |
| | $= \sec\theta\tan\theta - \int \sec\theta\left(1+\tan^2\theta\right)d\theta$ | | |
| | Hence $\int \sec\theta\tan^2\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int \sec\theta\,d\theta$ | | |
| | $= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln(\sec\theta + \tan\theta)$ | M1 | |
| | $[\ \ldots\ ]_0^{\arctan 2} = \frac{1}{2}\times\sqrt{5}\times 2 - \frac{1}{2}\ln\left(\sqrt{5}+2\right)$ | M1 A1 | |
| | $\int_0^4 \text{arsinh}\sqrt{x}\,dx = 4\,\text{arsinh}\,2 - \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right) = \frac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$ | A1 | |

[FP2 June 2007 Qn 7]

| | | |
|---|---|---|
| **63.** | (a) $\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ | |
| | Third row $1-3=-\lambda \quad \Rightarrow \quad \lambda = 2$ | M1 A1 **(2)** |
| | (b) $\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-p \\ q+4 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ | M1 A1 |
| | First row $4-p=0 \quad \Rightarrow \quad p=4$ Method for either | M1 |
| | Second row $q+4=2 \quad \Rightarrow \quad q=-2$ Both correct | A1 ft **(4)** |
| | (c) $\begin{pmatrix} 3 & 4 & 4 \\ -1 & -2 & -4 \\ 1 & 1 & 3 \end{pmatrix} . \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$ | |
| | $3l+4m+4n=10$<br>$-l-2m-4n=-4$<br>$l+m+3n=3$ Obtaining 3 linear equations | M1 |
| | $2l+2m=6$<br>$3l+2m=8$ Reducing to a pair of equations and solving for one variable | M1 |
| | $l=2, m=1, n=0$ Solving for all three variables. | M1 A1 **(4)**<br>**[10]** |
| | Alternative to (c) | |
| | $\mathbf{A}^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 8 & 8 \\ -1 & -5 & -8 \\ -1 & -1 & 2 \end{pmatrix}$ | M1 M1 |
| | $\frac{1}{6}\begin{pmatrix} 2 & 8 & 8 \\ -1 & -5 & -8 \\ -1 & -1 & 2 \end{pmatrix} . \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ | M1 A1 **(4)** |

[FP3 June 2007 Qn 3]

| | | |
|---|---|---|
| **64.** | (a) $\overrightarrow{AB} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}, \quad \overrightarrow{AC} = 4\mathbf{j} + 2\mathbf{k}$ any two | B1 |
| | $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 3 \\ 0 & 4 & 2 \end{vmatrix} = -6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ | M1 A1 A1 |
| | Give A1 for any two components correct or the negative of the correct answer. | **(4)** |
| | (b) Cartesian equation has form $3x - y + 2z = p$ | |
| | $(2, -1, 0) \;\Rightarrow\; 6 + 1 = p$ or use of another point | M1 |
| | $3x - y + 2z = 7$ ∗ or any multiple | A1 **(2)** |
| | (c) Parametric form of line is $\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ or equivalent form | M1 A1 |
| | Substituting into equation of plane | |
| | $3(5 + 2\lambda) - (5 - \lambda) + 2(3 - 2\lambda) = 7$ | M1 |
| | Leading to $\lambda = -3$ | A1 |
| | $T : (-1, 8, 9)$ | A1 **(5)** |
| | (d) $\overrightarrow{AT} = -3\mathbf{i} + 9\mathbf{j} + 9\mathbf{k}, \quad \overrightarrow{BT} = -2\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ both | M1 |
| | These are parallel and hence *A*, *B* and *T* are collinear ∗ (by the axiom of parallels) | M1 A1 **(3)** **[14]** |
| | Alternative to (d) | |
| | The equation of *AB*: $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mu(-\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ or equivalent | |
| | $\mathbf{i}$: $-1 = 2 - \mu \;\Rightarrow\; \mu = 3$ | M1 |
| | $\mu = 3 \;\Rightarrow\; \overrightarrow{OT} = -\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$ | M1 |
| | Hence *A*, *B* and *T* are collinear ∗ cso | A1 **(3)** |
| | Note: Column vectors or bold-faced vectors may be used at any stage. | |

[FP3 June 2007 Qn 7]

| | | | |
|---|---|---|---|
| **65.** | $\frac{d}{dx}(\ln(\tanh x)) = \frac{\text{sech}^2 x}{\tanh x}$ | M1 A1 | |
| | $= \frac{1}{\sinh x \cosh x} = \frac{2}{\sinh 2x} = 2\,\text{cosech}\, 2x$ (*) | M1 A1 | (4) |
| | | | **4** |
| | **Notes**<br>**1M1** Any valid differentiation attempt including $\ln(e^x - e^{-x}) - \ln(e^x + e^{-x})$<br>**1A1** c.a.o. (o.e e.g. $\frac{\cosh x}{\sinh x} - \frac{\sinh x}{\cosh x}$ )<br>**2M1** Proceeding to a hyperbolic expression in $2x$<br>**2A1** c.s.o. | | |

[FP2 June 2008 Qn 1]

| | | | |
|---|---|---|---|
| **66.** | $8\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^x - e^{-x}}{2}\right) = 13$ | B1 | |
| | $4e^x + 4e^{-x} - 2e^x + 2e^{-x} = 13$ | | |
| | $2e^{2x} - 13e^x + 6 = 0$ (or equiv.) | M1 A1 | |
| | $(2e^x - 1)(e^x - 6) = 0$ | | |
| | $e^x = \frac{1}{2}, \quad e^x = 6$ | M1 A1ft | |
| | $x = \ln\frac{1}{2}$ (or $-\ln 2$), $\quad x = \ln 6$ | A1 | (6) |
| | | | **6** |
| | **Notes**<br>**B1** Correctly substituting exponentials for all hyperbolics<br>**1M1** To a three term quadratic in $e^x$<br>**1A1** c.a.o. (o.e.)<br>**2M1** Solving their equation to $e^x =$<br>**2A1ft** f.t. their equation.<br>**3A1** c.a.o. | | |

[FP2 June 2008 Qn 2]

| | | |
|---|---|---|
| **67.** | $\int \frac{3}{\sqrt{x^2-9}}\,dx + \int \frac{x}{\sqrt{x^2-9}}\,dx$ | B1 |
| | $= \left[3\operatorname{arcosh}\frac{x}{3} + \sqrt{x^2-9}\right]$ | M1 A1 A1 |
| | $= \left[3\ln\left(\frac{x+\sqrt{x^2-9}}{(3)}\right) + \sqrt{x^2-9}\right]_5^6$ | |
| | $= \left(3\ln\left(\frac{6+\sqrt{27}}{3}\right) + \sqrt{27}\right) - \left(3\ln\left(\frac{5+4}{3}\right) + 4\right)$ | M1 A1 |
| | $= 3\ln\frac{6+\sqrt{27}}{9} + \sqrt{27} - 4 = 3\ln\frac{2+\sqrt{3}}{3} + 3\sqrt{3} - 4$ (*) | A1 (7) |
| | **Notes** | 7 |
| | **B1** Correctly changing to an integrable form.<br>**1M1** Complete attempt to integrate at least one bit.<br>**1A1** One term correct<br>**2A1** All correct<br>**2DM1** Substituting limits in all. **Must have got first M1**<br>**3A1** Correctly (no follow through)<br>**4A1** c.s.o. | |

[FP2 June 2008 Qn 3]

| | | | |
|---|---|---|---|
| **68.** | (a) $\frac{dy}{dx} = \frac{3x^2}{\sqrt{1+x^6}}$ , At $x = \sqrt{2}$ $\frac{dy}{dx} = \frac{6}{3} = 2$ | M1 A1, A1 | |
| | $y - \text{arsinh}\,(2\sqrt{2}) = 2(x - \sqrt{2})$ | M1 | |
| | $y = 2x - 2\sqrt{2} + \ln\left(3 + 2\sqrt{2}\right)$ (*) | A1 | (5) |
| | (b) $\frac{3a^2}{\sqrt{1+a^6}} = 2$ $9a^4 = 4(1 + a^6)$ | M1 A1 | |
| | $4a^6 - 9a^4 + 4 = 0$ $(a^2 - 2)(4a^4 - a^2 - 2) = 0$ | A1 | |
| | $a^2 = \frac{1 \pm \sqrt{1+32}}{8}$ $a = \sqrt{\frac{1+\sqrt{33}}{8}} \approx 0.92$ | M1 A1 | (5) |
| | | | **10** |
| | **Notes**<br>**(a)1M1** Attempt to differentiate need $\left(1 + x^6\right)^{-\frac{1}{2}}$ at least<br>**1A1** correct<br>**2A1** c.a.o.<br>**2M1** Substituting into straight line equation (linear). Must use $x = \sqrt{2}$<br>**3A1** c.s.o.<br>**(b)1M1** Their derivative = their gradient (condone $x$ throughout)<br>**2M1= A mark cao, any form**<br>**1A1** quartic cao<br>**3M1** Solving their quartic to '$a$' =<br>**2A1** c.a.o. (a.w.r.t. 0.92 to 2dp) | | |

[FP2 June 2008 Qn 4]

number

| | | | |
|---|---|---|---|
| **69.** | (a) $I_n = \int_0^{\pi} e^x \sin^n x\,dx = \left[e^x \sin^n x\right] - \int e^x n \sin^{n-1} x \cos x\,dx$ | M1 A1 | |
| | $\left[e^x \sin^n x - ne^x \sin^{n-1} x \cos x\right] + n\int e^x(-\sin^n x + (n-1)\cos x \sin^{n-2} x \cos x)\,dx$ | M1 A1 | |
| | $\left[e^x \sin^n x - ne^x \sin^{n-1} x \cos x\right]_0^{\pi} = 0$ | B1 | |
| | $I_n = -n\int e^x \sin^n x\,dx + n(n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$ | M1 | |
| | $I_n = -nI_n + n(n-1)I_{n-2} - n(n-1)I_n \qquad I_n = \frac{n(n-1)}{n^2+1}I_{n-2}$ (*) | M1 A1 | (8) |
| | (b) $I_4 = \frac{4\times 3}{17}I_2, \qquad = \frac{12}{17}\times\frac{2}{5}I_0$ | M1, A1 | |
| | $I_0 = \int_0^{\pi} e^x\,dx = \left[e^x\right]_0^{\pi} = \dots \qquad , \qquad I_4 = \frac{24}{85}\left(e^{\pi} - 1\right)$ | M1, A1 | (4) |
| | | **12** | |

**(a)1M1** Complete attempt to use parts once in the right direction need $\sin^{n-1} x$
**1A1** cao
**2M1** Attempt to use parts again with sensible choice of parts, not reversing. Need to be differentiating a product.
**2A1** cao
**1B1** both = 0 at some point. (doesn't need to be correct, must must =0)
**3DM1** $I_n$ = expressions in $\int e^x \sin^k x\,dx$ **Depends on 2nd M**
**4DM1** Expresssion in $I_n$ and $I_{n-2}$ to $I_n =$. **Depends on 3rd M**
**3A1** c.s.o.
**(b)1M1** $I_4$ in terms of $I_2$
**1A1** $I_4$ correctly in terms of $I_0$ [ o.e.]
**2M1** $\int e^x dx$
**2A1** c.a.o for $I_4$ .

[FP2 June 2008 Qn 5]

| | | | |
|---|---|---|---|
| **70.** | (a) $\int \cosh x \arctan(\sinh x)\,dx = \sinh x \arctan(\sinh x) - \int \sinh x \frac{\cosh x}{1+\sinh^2 x}\,dx$ | M1 A1 A1 | |
| | $= \sinh x \arctan(\sinh x) - \frac{1}{2}\ln(1+\sinh^2 x)\ (+C)$ | M1 A1 | (5) |
| | Or: $\ldots\ldots\ldots\ldots - \int \tanh x\,dx$ | | |
| | $= \sinh x \arctan(\sinh x) - \ln(\cosh x)\ (+C)$ M1 A1 | | |
| | <u>Alternative:</u> | | |
| | Let $t = \sinh x$, $\frac{dt}{dx} = \cosh x$, $\int \arctan t\,dt = t \arctan t - \int \frac{t}{1+t^2}\,dt$ M1 A1 A1 | | |
| | $= \ldots\ldots - \frac{1}{2}\ln(1+t^2)$ M1 | | |
| | $= \sinh x \arctan(\sinh x) - \frac{1}{2}\ln(1+\sinh^2 x)\ (+C)$ (or equiv.) A1 | | |
| | (b) $\frac{1}{10}\left[\sinh x \arctan(\sinh x) - \ln(\cosh x)\right]_0^2 = \ldots\ldots,$ 0.34 (*) | M1, A1 | (2) |
| | | | 7 |
| | (a) <u>Alternative:</u><br>Let $\tan t = \sinh x$, $\sec^2 t \frac{dt}{dx} = \cosh x$, $\int t \sec^2 t\,dt = t \tan t - \int \tan t\,dt$ M1 A1 A1<br>$= \ldots\ldots - \ln(\sec t)$ M1<br>$= \sinh x \arctan(\sinh x) - \ln\sqrt{1+\sinh^2 x}\ (+C)$ (or equiv.) A1<br><br>**Notes**<br>**(a)1M1** Complete attempt to use parts<br>**1A1** One term correct.<br>**2A1** All correct.<br>**2M1** All integration completed. Need a ln term.<br>**3A1** c.a.o. ( in $x$) o.e, any correct form, simplified or not<br>**(b)1M1** Use of limits 0 and 2 and 1/10.<br>**1A1** c.s.o. | | |

[FP2 June 2008 Qn 6]

| | | | |
|---|---|---|---|
| **71.** | (a) $\frac{2x}{16}-\frac{2y}{9}\frac{dy}{dx}=0$ $\left[\frac{dx}{dt}=4\sec t\tan t,\ \frac{dy}{dt}=3\sec^2 t\right]$ | M1 A1 | |
| | $\frac{dy}{dx}=\frac{9x}{16y}=\frac{36\sec t}{48\tan t}=\frac{3}{4\sin t}$ | M1 A1 | |
| | $y-3\tan t=\frac{-4\sin t}{3}(x-4\sec t)$ | M1 | |
| | $4x\sin t+3y=25\tan t$ (*) | A1 | (6) |
| | (b) Using $b^2=a^2(e^2-1)$: $ae=\sqrt{a^2+b^2}=5$ or $e=\frac{5}{4}$ | M1 A1 | |
| | $P$: $4\sec t=5$ $\cos t=\frac{4}{5}$ | M1 | |
| | Coordinates of $P$: $(4\sec t, 3\tan t)=\left(5,\frac{9}{4}\right)$ | M1 A1 | (5) |
| | (c) $R$: $x=\frac{25\tan t}{4\sin t}=\frac{125}{16}$ | M1 | |
| | Area of $PRS$: $\frac{1}{2}(SR\times SP)=\frac{1}{2}\times\left(\frac{125}{16}-5\right)\times\frac{9}{4}=\frac{405}{128}\left(=3\frac{21}{128}\right)$ | M1 A1 | (3) |
| | | **14** | |
| | **Notes**<br>**(a)1M1** Differentitating<br>**1A1** c.a.o.<br>**2M1** $\frac{dy}{dx}$ in terms of $t$.<br>**2A1** c.a.o.<br>**3M1** Substituting gradient of **normal** into straight line equation.<br>**3A1** c.s.o.<br>**(b)1M1** Use of $b^2=a^2(e^2-1)$<br>**1A1** c.a.o. for ae or for e<br>**2M1** Using $x$ coordinate of focus= $x$ coordinate of P, to get single term $f(t)$= constant. (**Allow recovery in (c)** )<br>**3M1** Substituting into P coordinates to a number for $x$ and for $y$.<br>**2A1** c.a.o.<br>**(c)1M1** Attempt to find $x$ coordinate of R.<br>**2M1** Substituting into correct template i.e. ½ x \|their $R_x$ - their $H_x$\| x their $P_y$<br>**1A1** c.a.o. 3 s.f. or better. | | |

[FP2 June 2008 Qn 7]

| | | | |
|---|---|---|---|
| **72.** | (a) | $\begin{pmatrix}1 & p & 2\\0 & 3 & q\\2 & p & 1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$  $\begin{pmatrix}1+2p+2\\6+q\\2+2p+1\end{pmatrix} = \begin{pmatrix}\lambda\\2\lambda\\\lambda\end{pmatrix}$ is M1 A1 (2 eqns implied)<br>$\begin{pmatrix}3+2p\\6+q\\3+2p\end{pmatrix} \Rightarrow 6+q = 2(3+2p)$ is M1 A1 (2 eqns, use of parameter implied)<br>$1+2p+2=\lambda$  $6+q=2\lambda$  M: Two equations, one in $p$, one in $q$<br>$\therefore\ 6+q=6+4p \Rightarrow q=4p$ (*) | <br><br><br>M1 A1<br>A1 (3) |
| | (b) | $\begin{vmatrix}-4 & p & 2\\0 & -2 & 4p\\2 & p & -4\end{vmatrix}=0$ or $\begin{vmatrix}1-\lambda & p & 2\\0 & 3-\lambda & 4p\\2 & p & 1-\lambda\end{vmatrix}=0$ (or with $q$ instead of $4p$)<br>$[-4(8-4p^2)-p(0-8p)+2(0+4)=0]$  $p^2=1$ or $pq=4$<br>$p<0$  $p=-1$  $q=-4$  M: Use $q=4p$ to find value of $p$ and of $q$<br>A1: Positive values must be rejected | M1<br><br>A1<br>dM1 A1 (4) |
| | (c) | $-4x-y+2z=0,\ -2y-4z=0,\ 2x-y-4z=0$ Any 2 eqns, with value of $p$<br>$2x=-y=2z$  (or 2 separate equations)<br>E.vector is $k\begin{pmatrix}1\\-2\\1\end{pmatrix}$  (Any non-zero value of $k$) | M1<br>M1<br>A1 (3)<br><br>**(10)** |
| | | (a) Assuming a value for $\lambda$, e.g. $\lambda=1$, gives M1 A0 A0.<br>(a) Assuming result and working 'backwards':<br>$\begin{pmatrix}1 & p & 2\\0 & 3 & 4p\\2 & p & 1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}3+2p\\6+4p\\3+2p\end{pmatrix} = (3+2p)\begin{pmatrix}1\\2\\1\end{pmatrix}$, gives M1 A0 A0<br>(b) <u>Alternative</u>:<br>$\begin{pmatrix}1 & p & 2\\0 & 3 & 4p\\2 & p & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 5\begin{pmatrix}x\\y\\z\end{pmatrix}$ or $\begin{pmatrix}-4 & p & 2\\0 & -2 & 4p\\2 & p & -4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ (or $q$ instead of $4p$)<br>$x+py+2z=5x,\ 3y+4pz=5y,\ 2x+py+z=5z$<br>$py+2z=4x$ (i), $2pz=y$ (ii), $2x+py=4z$ (iii)<br>From (i) and (iii)  $py=2z$<br>From (ii)  $p^2=1$  (or equiv. in terms of $p$ and/or $q$)<br>$p<0, p=-1, q=-4$  A1: Positive values must be rejected<br>(b) Using the eigenvector $\begin{pmatrix}1\\2\\1\end{pmatrix}$ scores <u>no marks</u> in this part. | <br><br><br><br>M1<br><br><br><br>A1<br>dM1 A1 |

[FP3 June 2008 Qn 2]

| | | | |
|---|---|---|---|
| **73.** | (a) | $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\overrightarrow{PR} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ | B1 |
| | | $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ | M1 A1 (3) |
| | (b) | $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = (\mathbf{i} - \mathbf{k}) \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k})$ [may use $\overrightarrow{OQ}$ or $\overrightarrow{OR}$] | M1 |
| | | $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 4$ o.e. ft from (a) | A1ft (2) |
| | (c) | $3x + y - z = 4$ (i), $x - 2y - 5z = 6$ (ii) | |
| | | (i) × 2 + (ii) $7x - 7z = 14$, $x = z + 2$ (M: Eliminate one variable) | M1 |
| | | In (ii) $z + 2 - 2y - 5z = 6$, $y + 2 = -2z$ (M: Substitute back) | M1 |
| | | $\therefore x = z + 2$ and $y + 2 = -2z$ o.e. ($y = 2 - 2x$) (Two correct '3-term' equations) | A1 |
| | | $\dfrac{x-2}{(1)} = \dfrac{y+2}{-2} = \dfrac{z}{(1)}$ o.e. (M: Form cartesian equations) | M1 A1 (5) |
| | (d) | Writing down direction vector of $\overrightarrow{PS}$ from part (c). | M1 |
| | | $\overrightarrow{QR} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} = \overrightarrow{PS}$ $\therefore PS$ // QR (or cross-product = 0) | A1 (2) |
| | (e) | $\overrightarrow{PT} = 4\mathbf{i} + 2\mathbf{j}$ (or $\overrightarrow{QT} = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ or $\overrightarrow{RT} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$) | |
| | | Volume $= \frac{1}{3}\left|\overrightarrow{PQ} \times \overrightarrow{PR} . \overrightarrow{PT}\right| = \frac{1}{3}\left|(3\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (4\mathbf{i} + 2\mathbf{j})\right|$ ft from (a) | M1 A1ft |
| | | (Instead of $\overrightarrow{PQ} \times \overrightarrow{PR}$, it could be $\overrightarrow{PQ} \times \overrightarrow{QR}$ or $\overrightarrow{PR} \times \overrightarrow{QR}$) | |
| | | $= \frac{1}{3}(12 + 2)$ | |
| | | $= 4\frac{2}{3}$ o.e. | A1 (3) |
| | | | **(15)** |
| | | (a) If both vectors are 'reversed', B0 M1 A1 is possible | |
| | | (c) <u>Alternative</u>: | |
| | | Direction of line: $\begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 7\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ M2 A1 | |
| | | Through $P$ $(1, 0, -1)$: $\dfrac{x-1}{1} = \dfrac{y}{-2} = \dfrac{z+1}{1}$ M1 A1 | |
| | | (e) <u>Alternative</u>: | |
| | | $\frac{1}{3}\begin{vmatrix} 4 & 2 & 0 \\ 1 & -1 & 2 \\ 2 & -3 & 3 \end{vmatrix}$ gives M1 A1 directly. Here ft from 1st line of part (a). | |
| | | <u>Special case</u>: | |
| | | $\frac{1}{6}$ or $\frac{1}{2}$ instead of $\frac{1}{3}$, but method otherwise correct: M1 A0 A0 | |

[FP3 June 2008 Qn 7]